Welcome to CS103!

Hello from Cynthia Bailey Lee!

“Like a cat lady, but for chickens.”
(a friend’s description of me)

Hello from Alex Aiken!

Fun fact: has traveled
with family to Reunion Island,
off the coast of Madagascar!

Are there “laws of physics”
in computer science?

Key Questions in CS103

- What problems can you solve with a computer?
  - ***Computability Theory***
- Why are some problems harder to solve than others?
  - ***Complexity Theory***
- How can we be certain in our answers to these questions?
  - ***Discrete Mathematics***

Course Website

https://cs103.stanford.edu

Almost all course content will be hosted here.

Let’s go to the website and check out the syllabus!

Problem Set 0

- Your first assignment, Problem Set 0, goes out later today. It’s due Friday at 2:30pm PT.
- This assignment requires you to:
  - Make sure the QT software you need for the class is working on your computer.
  - Make sure you’ve absorbed the course Honor Code.
  - Make sure you’ll be able to attend all our exams as scheduled (while it’s not too late to switch out of this class if not).

We've got a big journey ahead of us.

Let's get started!

Introduction to Set Theory

A ***set*** is an unordered collection of distinct objects, which may be anything, including other sets.

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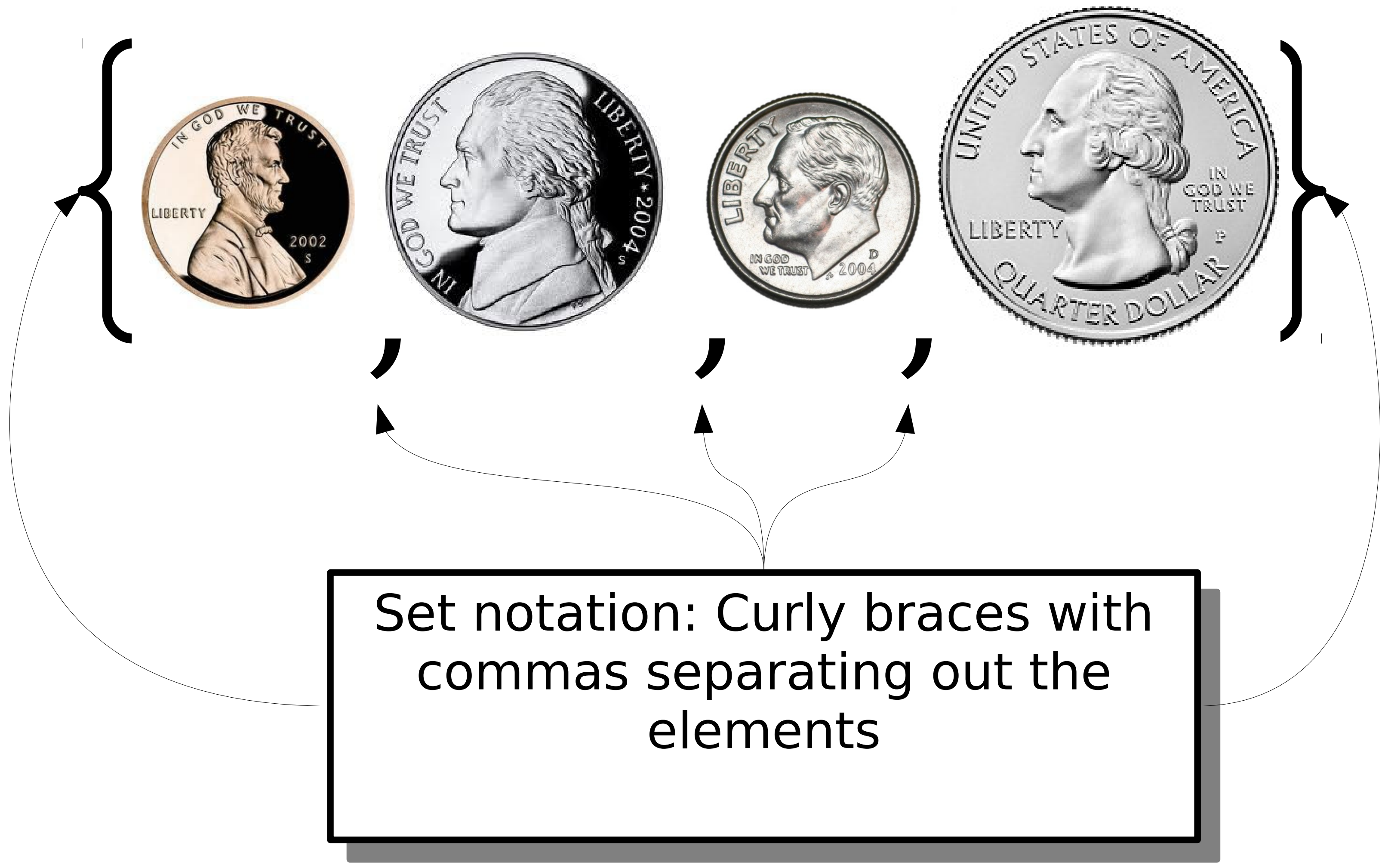
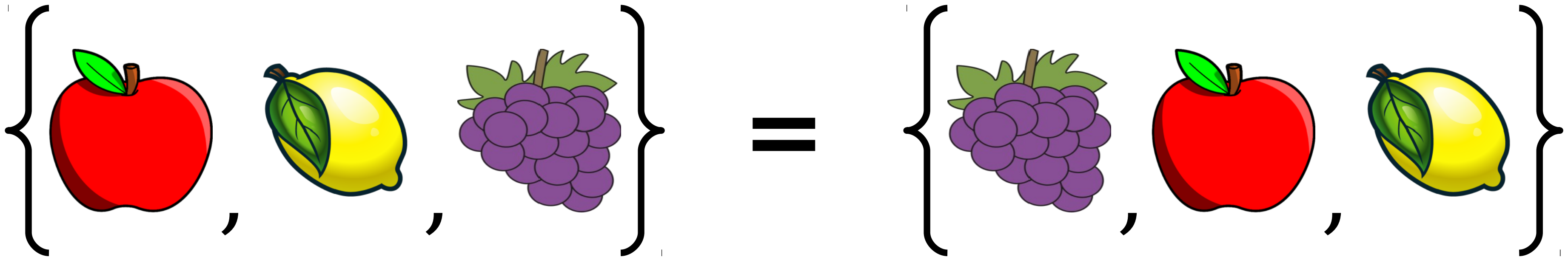
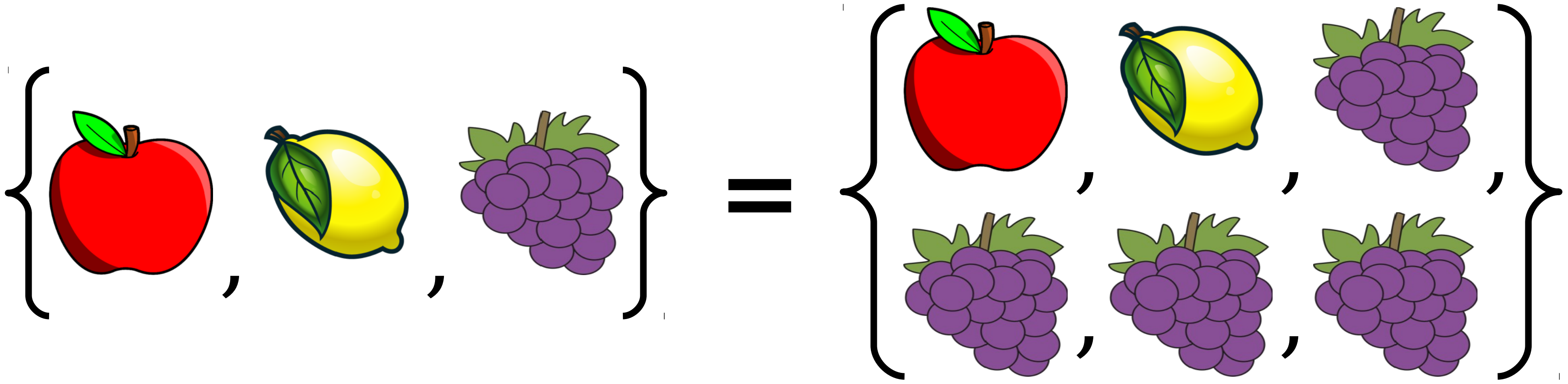


A ***set*** is an unordered collection of distinct objects, which may be anything, including other sets.



These are two different descriptions of exactly the same set.

Two sets are equal when they have the same contents, ignoring order.



These are also two different descriptions of exactly the same set.
(But please use the description without duplication :-))

Sets cannot contain duplicate elements.
Any repeated elements are ignored.

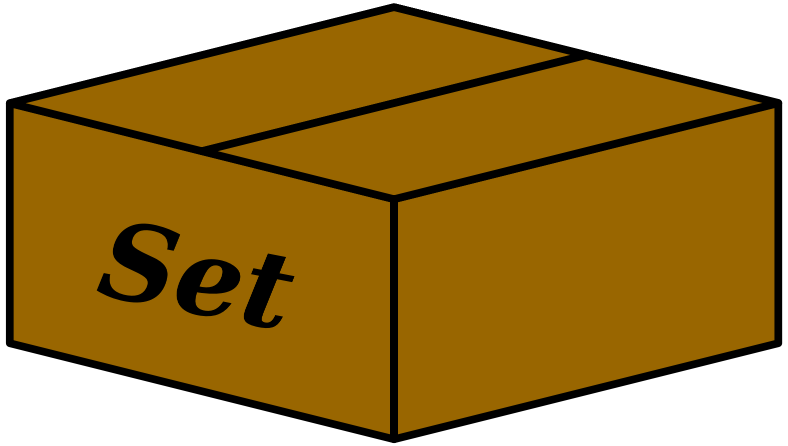
The objects that make up a set are called the ***elements*** of that set.

This symbol means "is an element of."

The objects that make up a set are called the ***elements*** of that set.

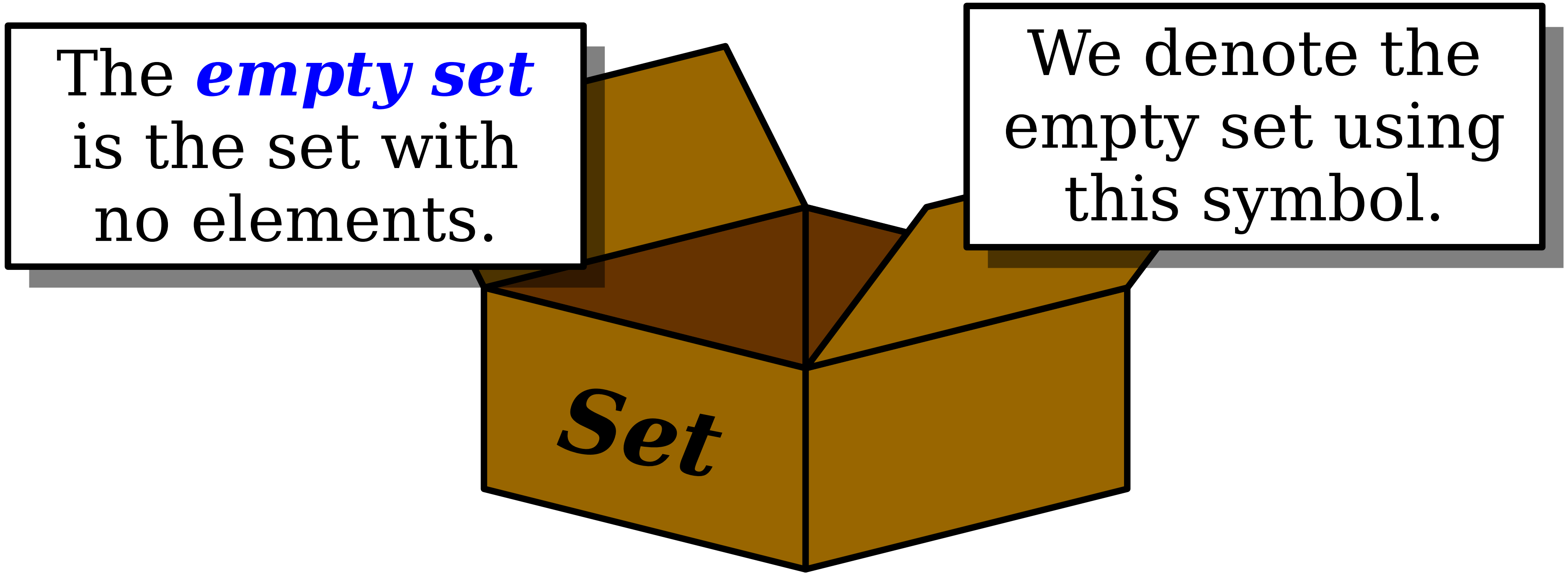
This symbol means "is not an element of."

The objects that make up a set are called the ***elements*** of that set.



Sets can contain any number of elements.

$$\{\ \} = \varnothing$$

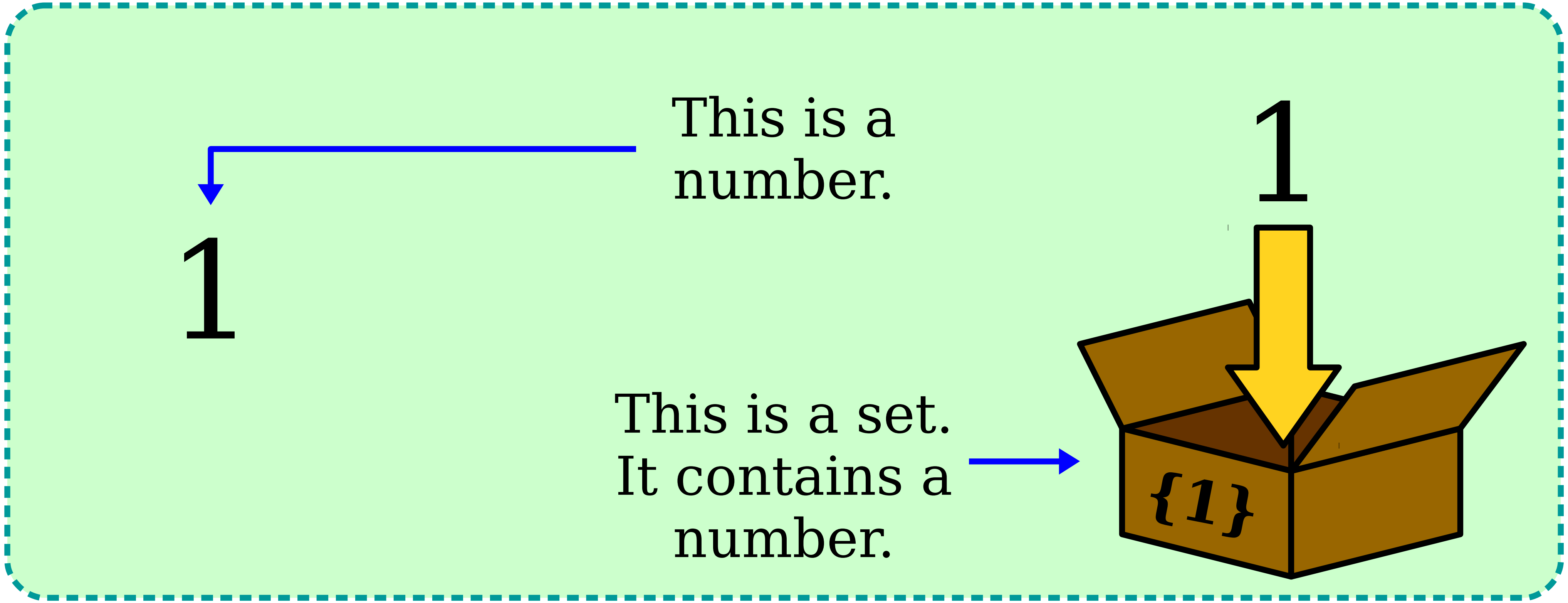


Sets can contain any number of elements.

$$1 \stackrel{?}{=} \{1\}$$

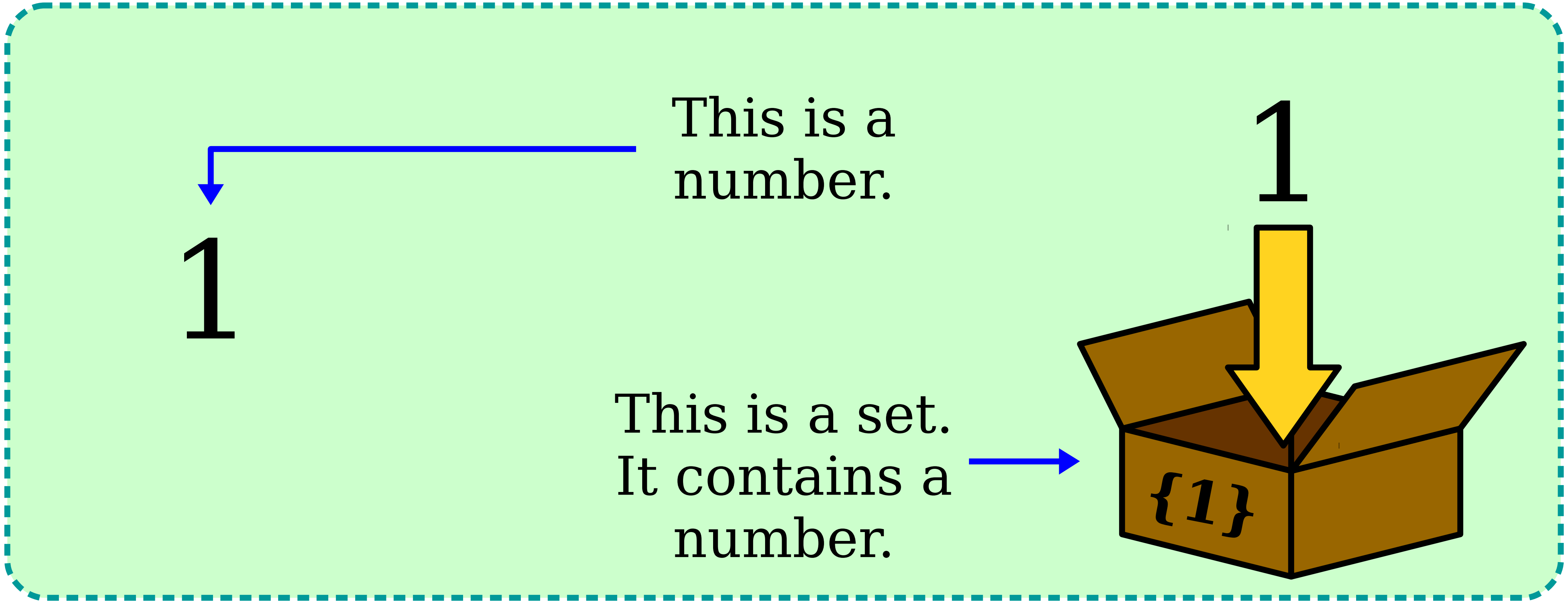
Question: Are these objects equal?

$$1 \stackrel{?}{=} \{1\}$$



Question: Are these objects equal?

$$1 \neq \{1\}$$



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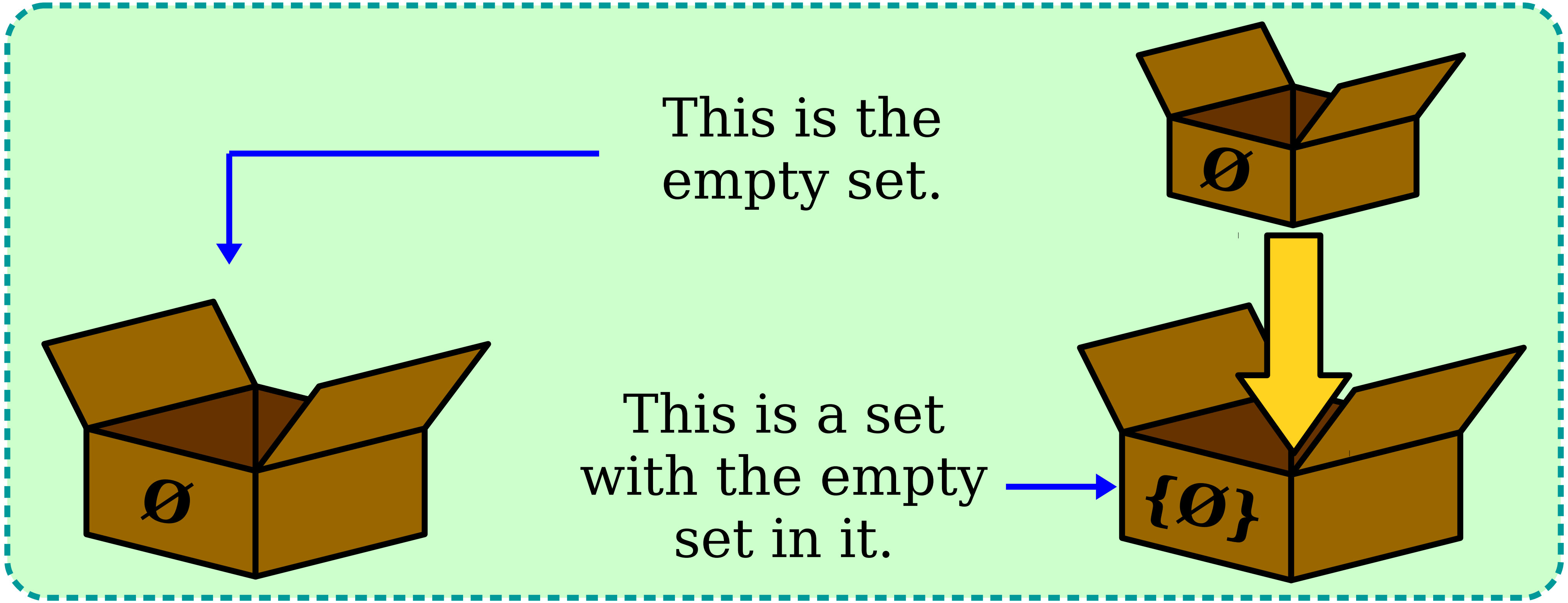

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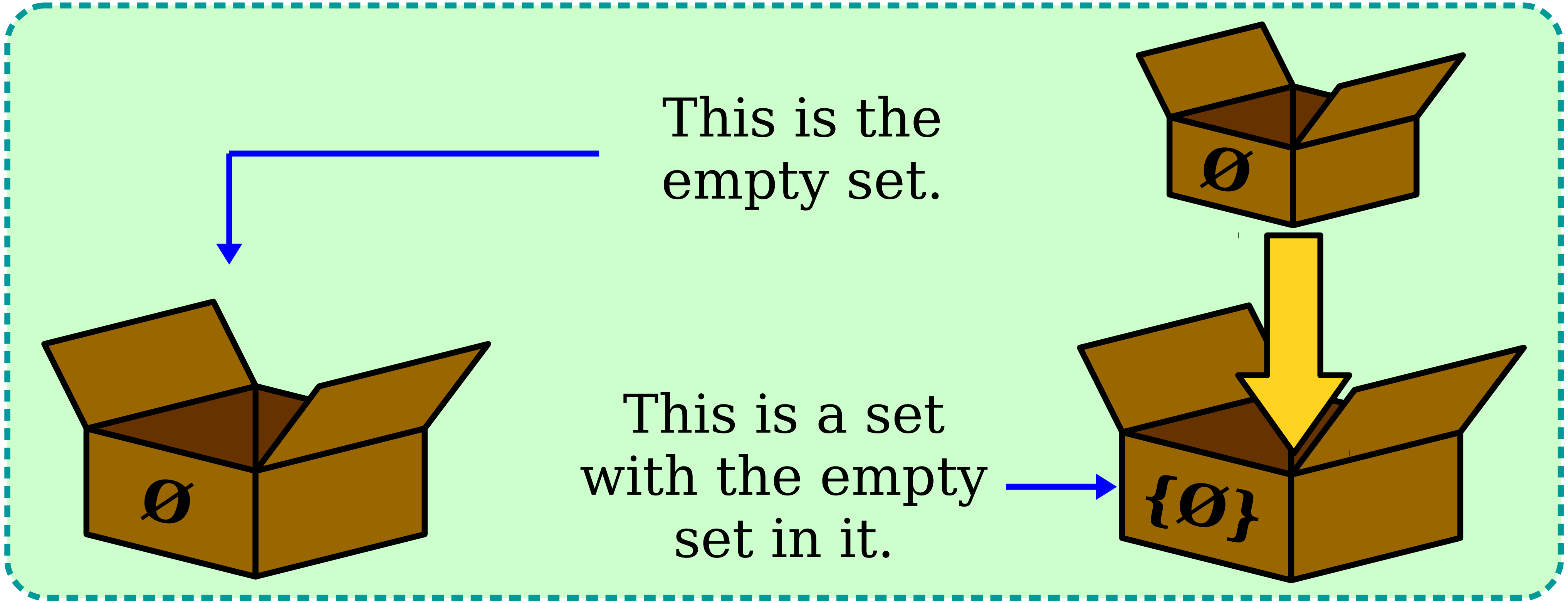
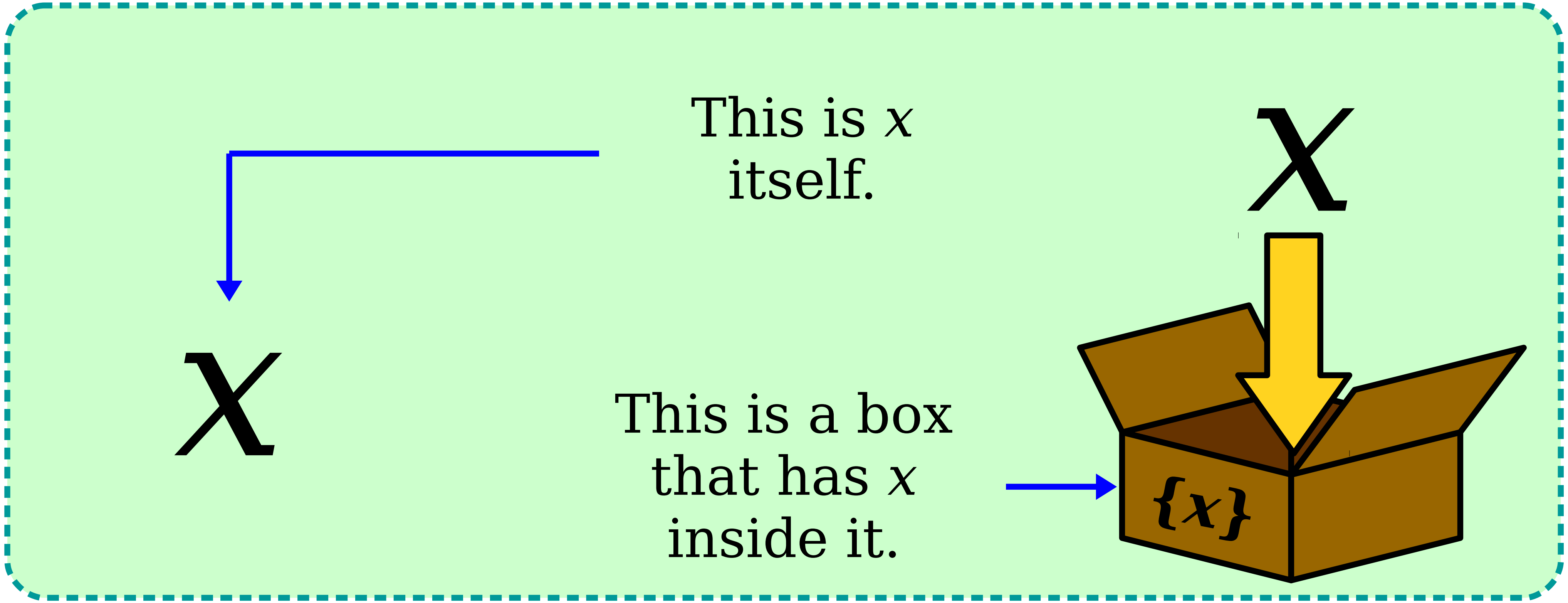


Question: Are these objects equal?

$$x \neq \{x\}$$



No object *x* is equal to the set containing *x*.

Infinite Sets

- Some sets contain *infinitely many* elements!
- The set ℕ = { 0, 1, 2, 3, …} is the set of all the ***natural numbers***.
  - Some mathematicians don't include zero; in this class, assume that 0 is a natural number.
- The set ℤ = { …, -2, -1, 0, 1, 2, … } is the set of all the ***integers***.
  - Z is from German "Zahlen."
- The set ℝ is the set of all ***real numbers***.
  - $e \in \mathbb{R}$, $\pi \in \mathbb{R}$, $4 \in \mathbb{R}$, etc.

Describing Complex Sets

- Here are some English descriptions of infinite sets:

  “The set of all even natural numbers.”

  “The set of all real numbers less than 137.”

  “The set of all negative integers.”

- To describe complex sets like these mathematically, we'll use ***set-builder notation***.

Even Natural Numbers

$$\{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even }\}$$

Even Natural Numbers

$$\{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

Even Natural Numbers

$$\{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even } \}$$

The set of all n

Even Natural Numbers

$$\{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even}\ \}$$

The set of all n

where

Even Natural Numbers

$$\{\, n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \,\}$$

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The set of all n

where

n is a natural number

and n is even

$$\{\, 0, 2, 4, 6, 8, 10, 12, 14, 16, \ldots \,\}$$

Set Builder Notation

- A set may be specified in ***set-builder notation***:

$$\{\ x \mid \textit{some property } x \textit{ satisfies}\ \}$$

$$\{\ x \in S \mid \textit{some property } x \textit{ satisfies}\ \}$$

- For example:

$\{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even}\ \}$

$\{\ C \mid C \text{ is a set of US currency}\ \}$

$\{\ r \in \mathbb{R} \mid r < 3\ \}$

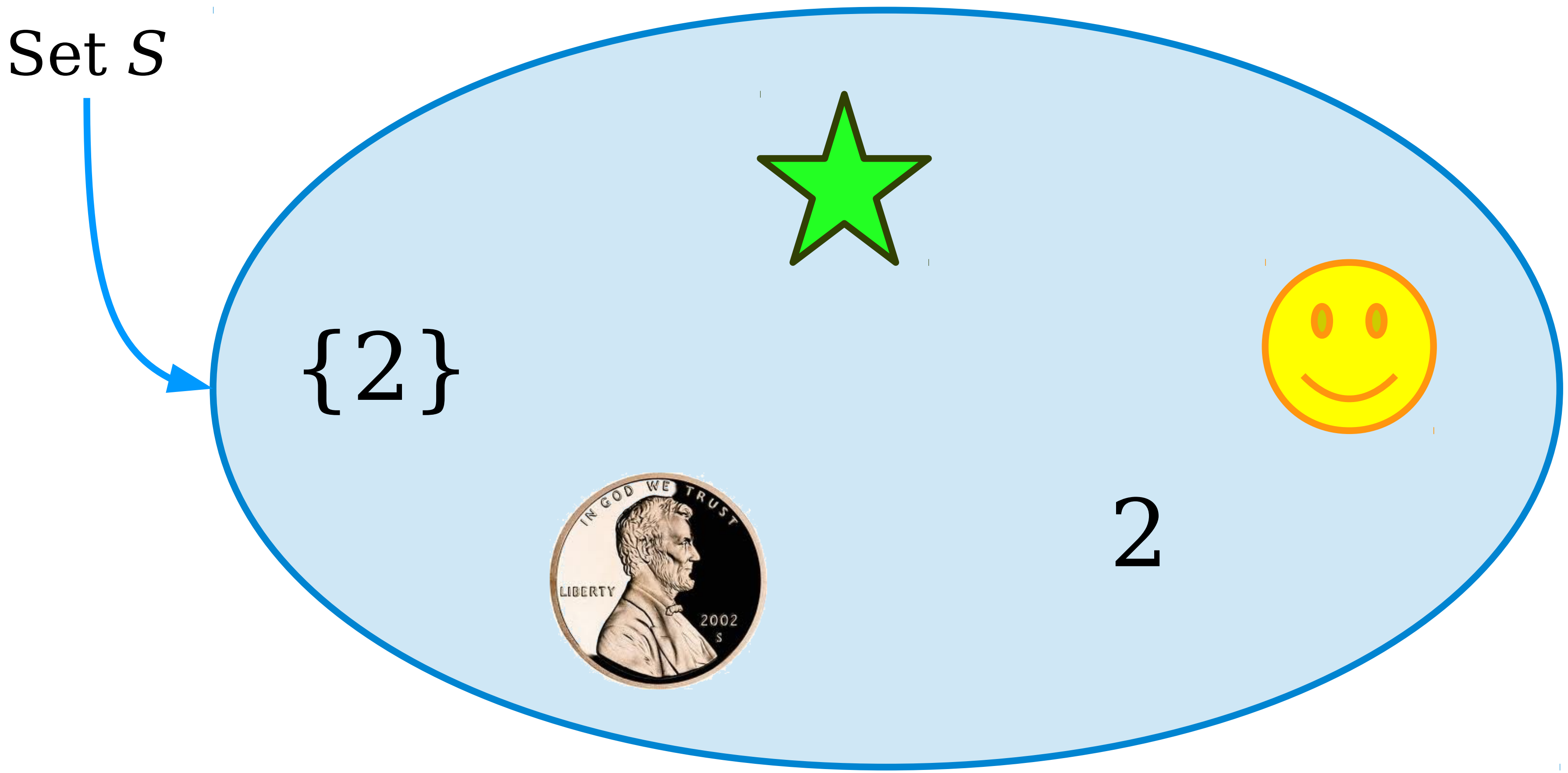
$\{\ n \in \mathbb{N} \mid n < 3\ \}$ **(the set {0, 1, 2})**

Subsets and Power Sets

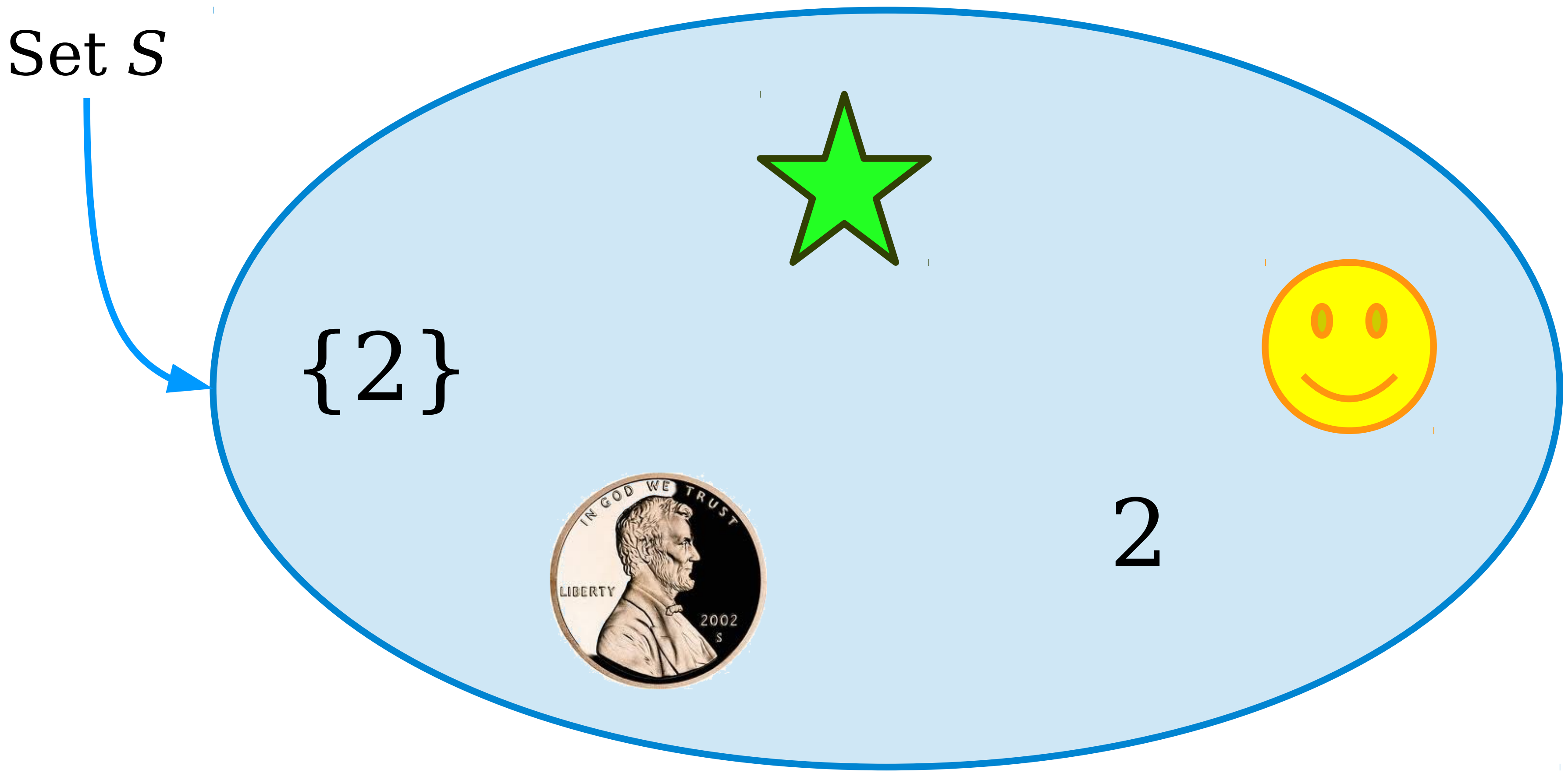
Subsets

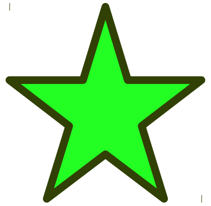
- A set S is called a ***subset*** of a set T (denoted $\boldsymbol{S \subseteq T}$) if all elements of S are also elements of T.
- Examples:
  - $\{ 1, 2, 3 \} \subseteq \{ 1, 2, 3, 4 \}$
  - $\{ b, c \} \subseteq \{ a, b, c, d \}$
  - $\{ \text{H}, \text{He}, \text{Li} \} \subseteq \{ \text{H}, \text{He}, \text{Li} \}$
  - $\mathbb{N} \subseteq \mathbb{Z}$ (*every natural number is an integer*)
  - $\mathbb{Z} \subseteq \mathbb{R}$ (*every integer is a real number*)

Subsets and Elements

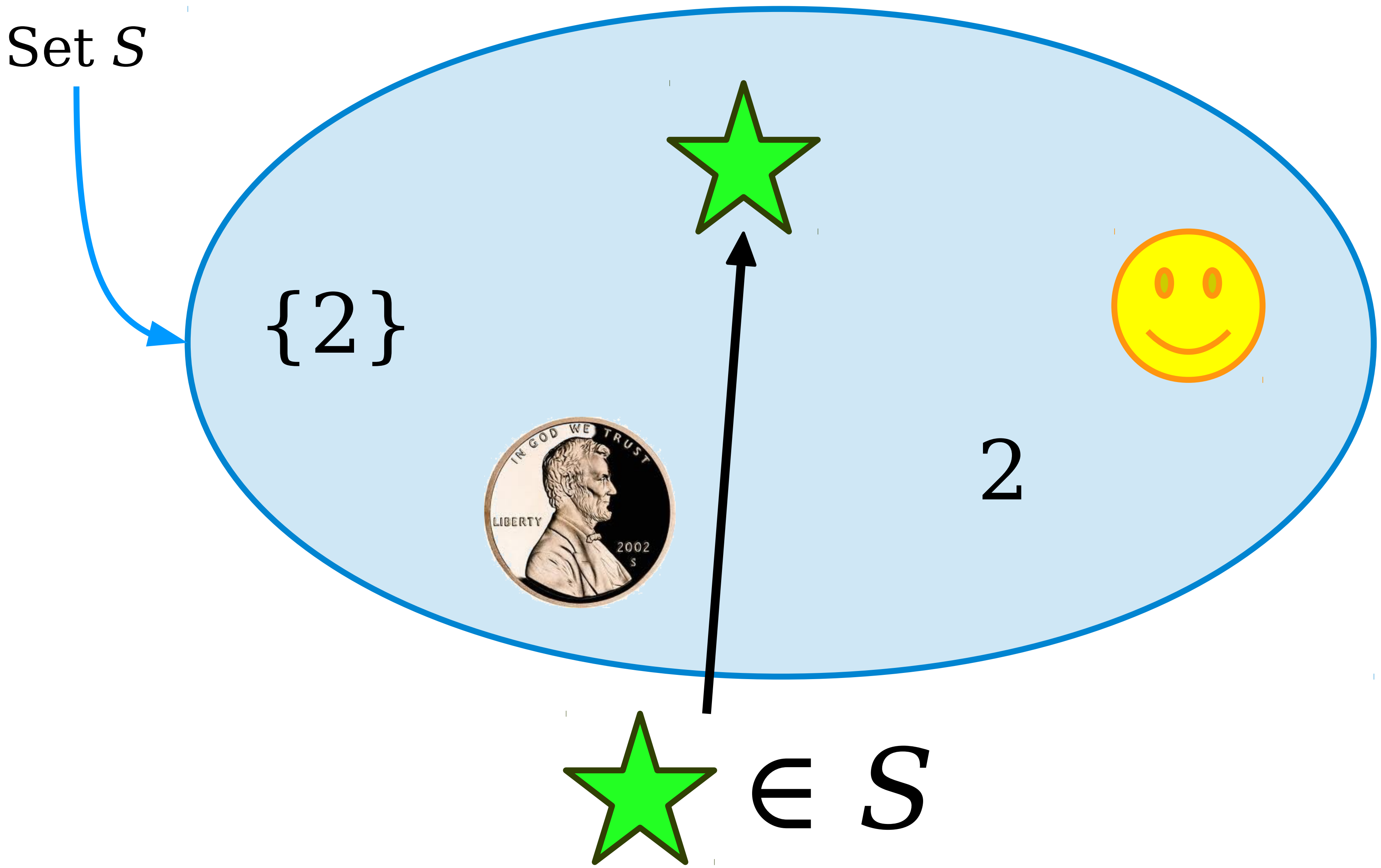


Subsets and Elements



 $\in S$

Subsets and Elements



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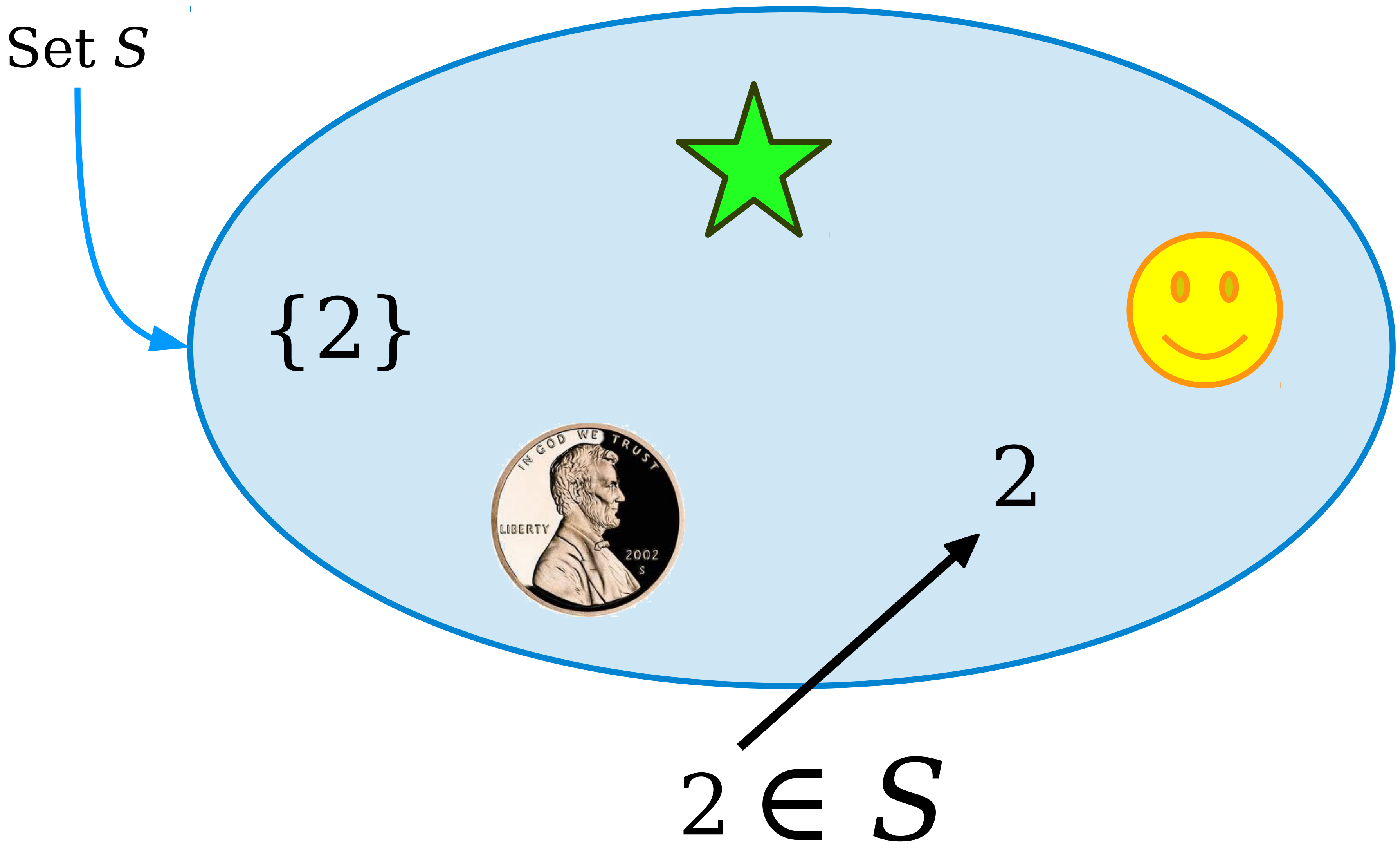

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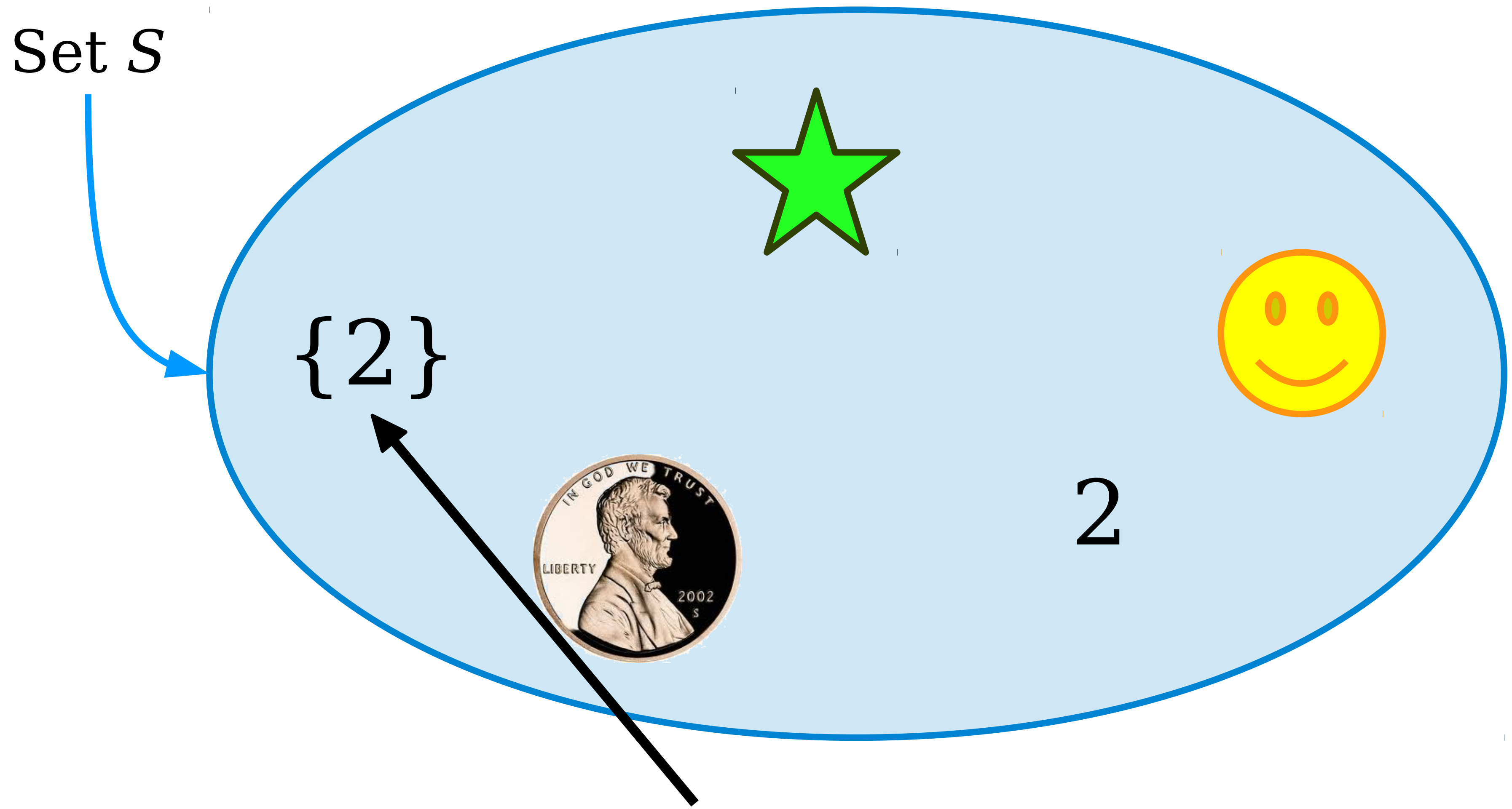
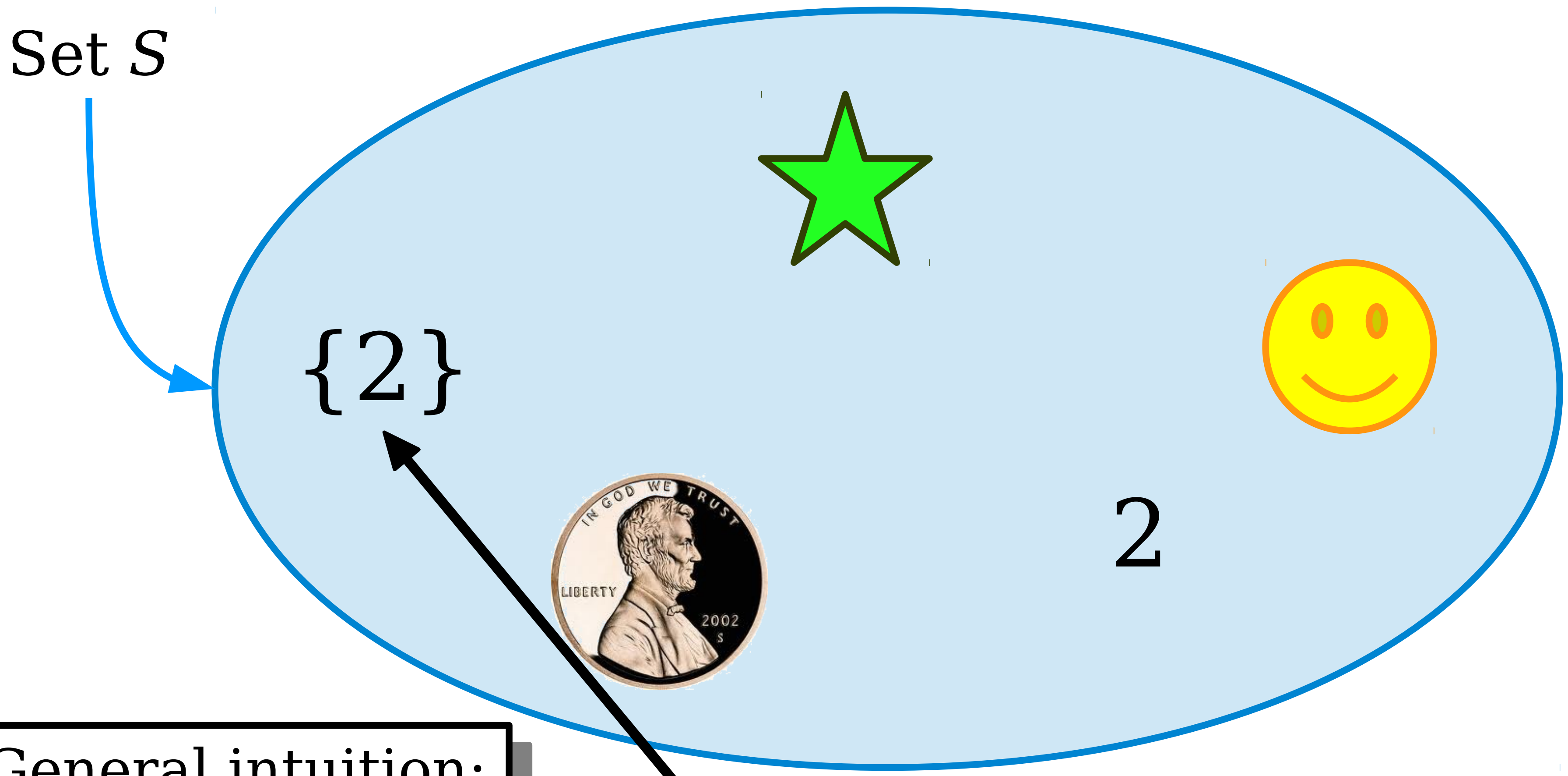


$\{2\} \in S$

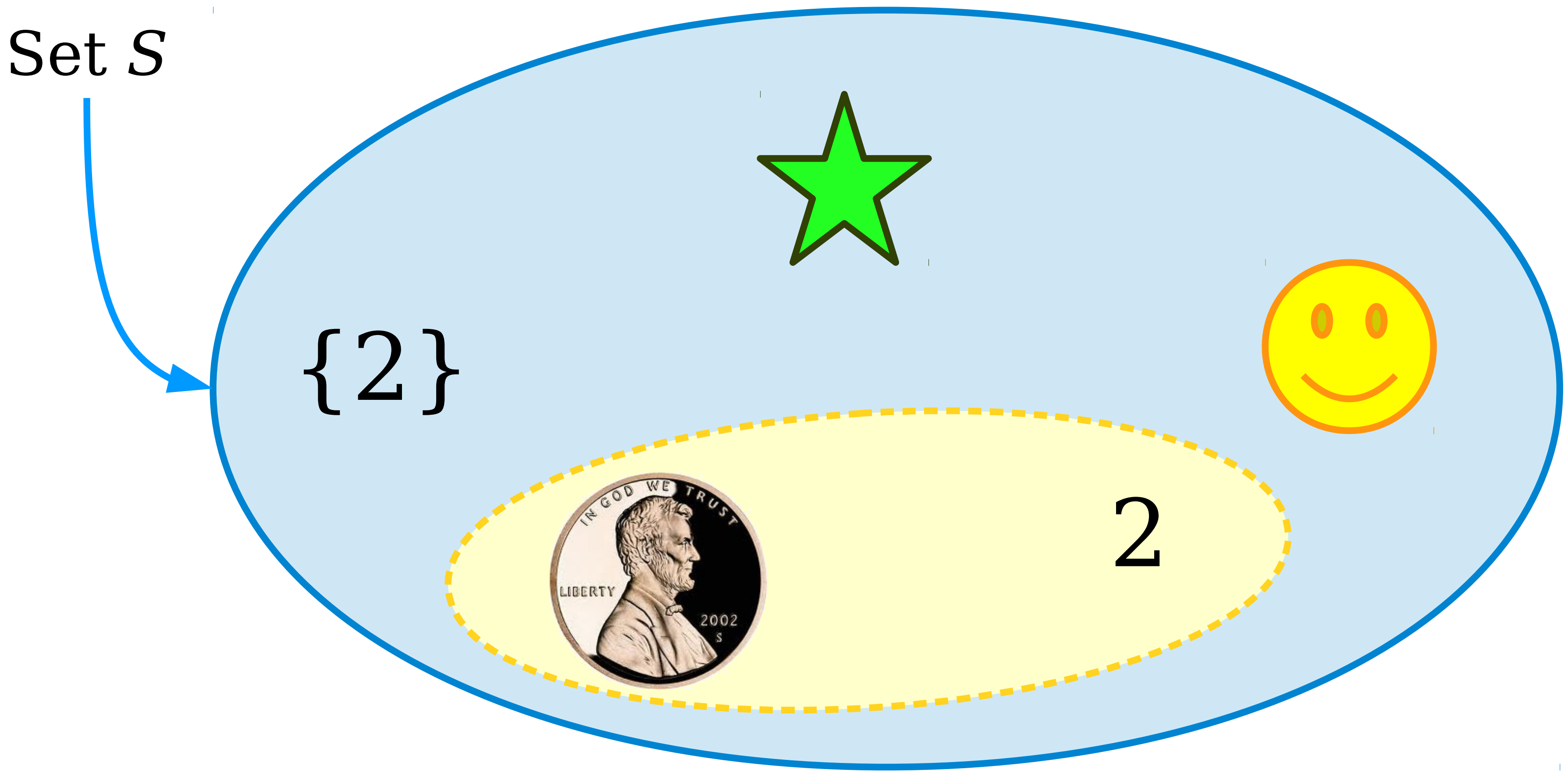
Subsets and Elements



$\{2\} \in S$

General intuition: $x \in S$ means you can ***point at x inside of S***.

Subsets and Elements



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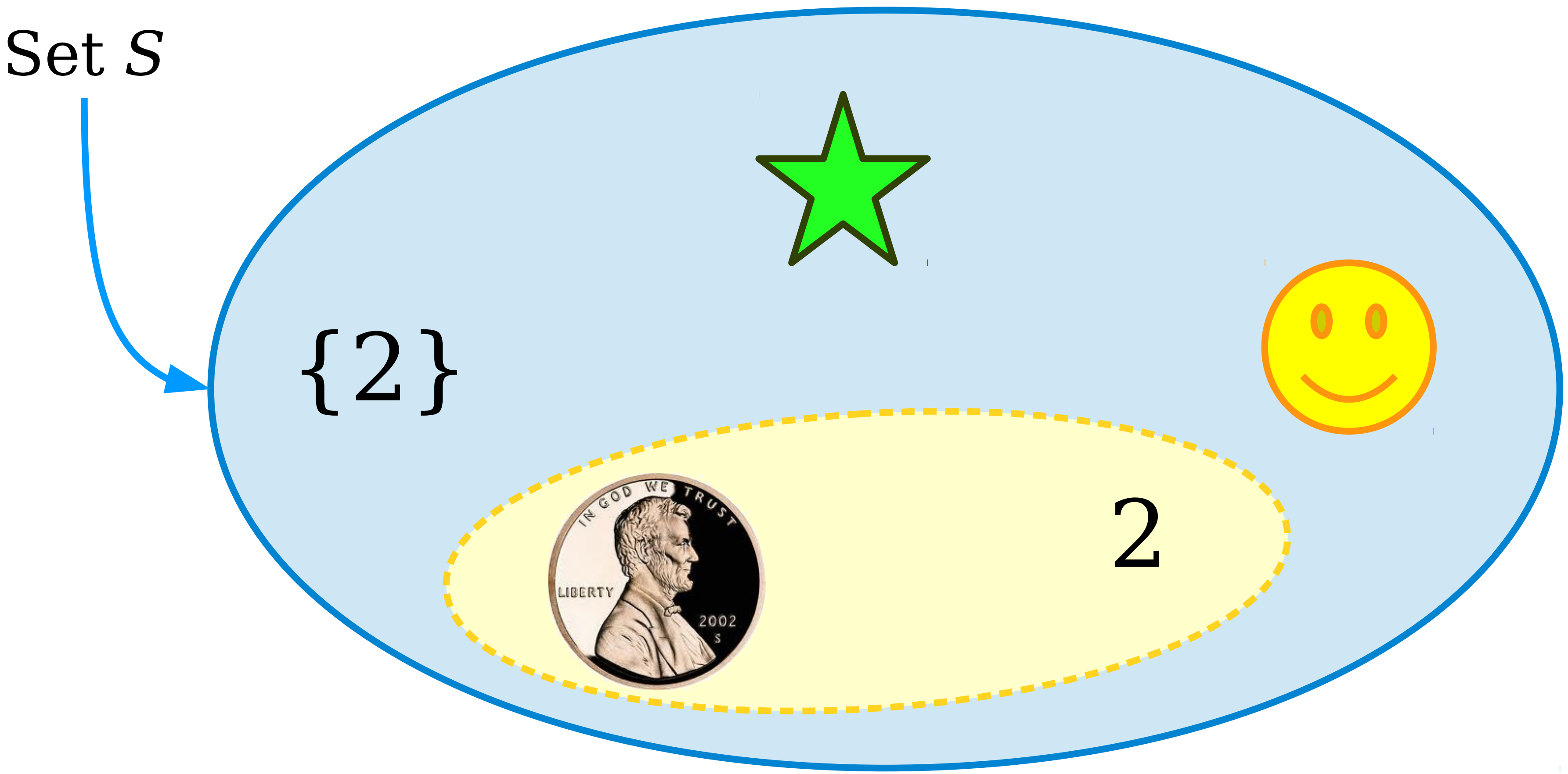
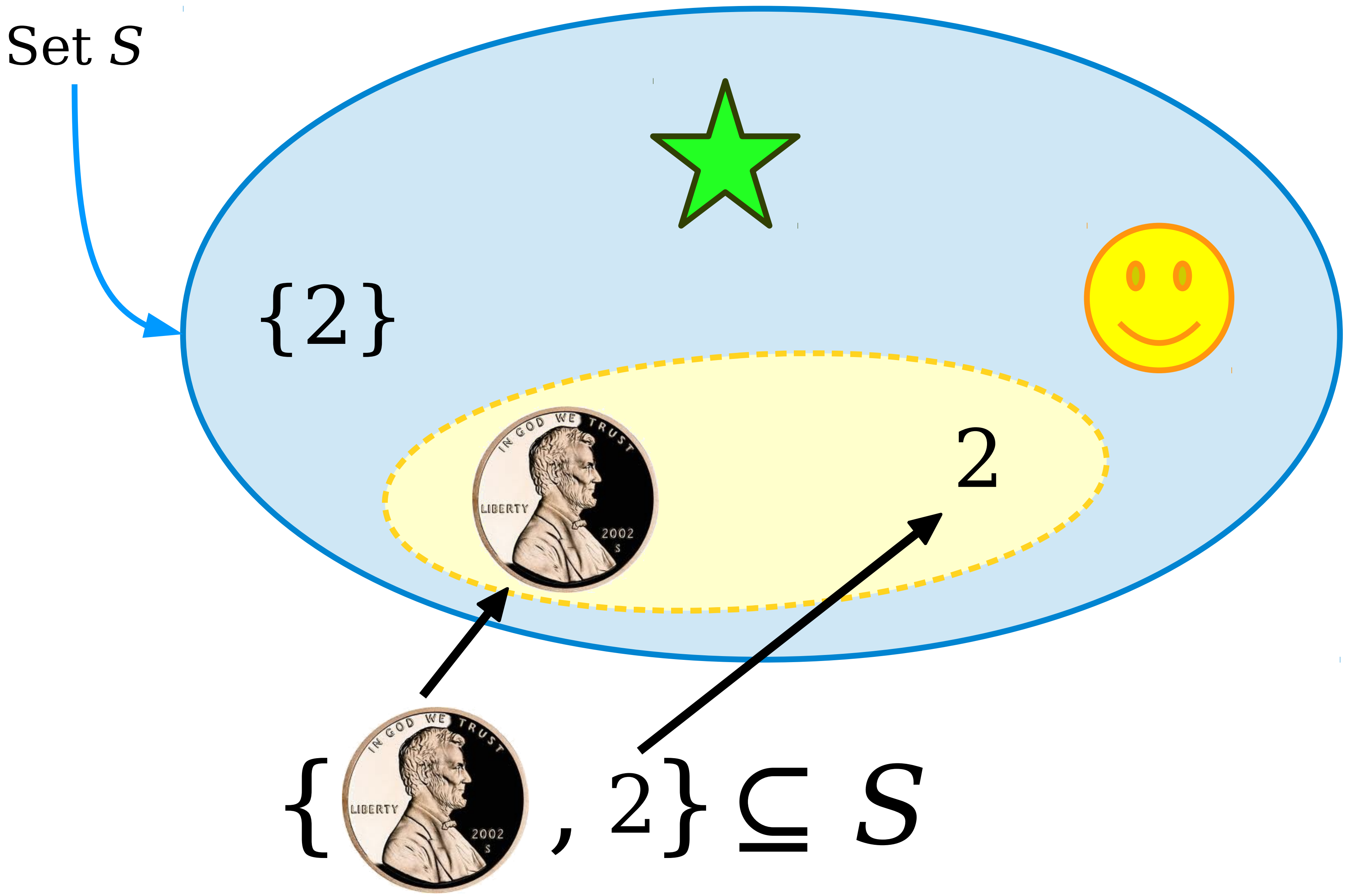


$\{$  $, 2\} \subseteq S$

Subsets and Elements



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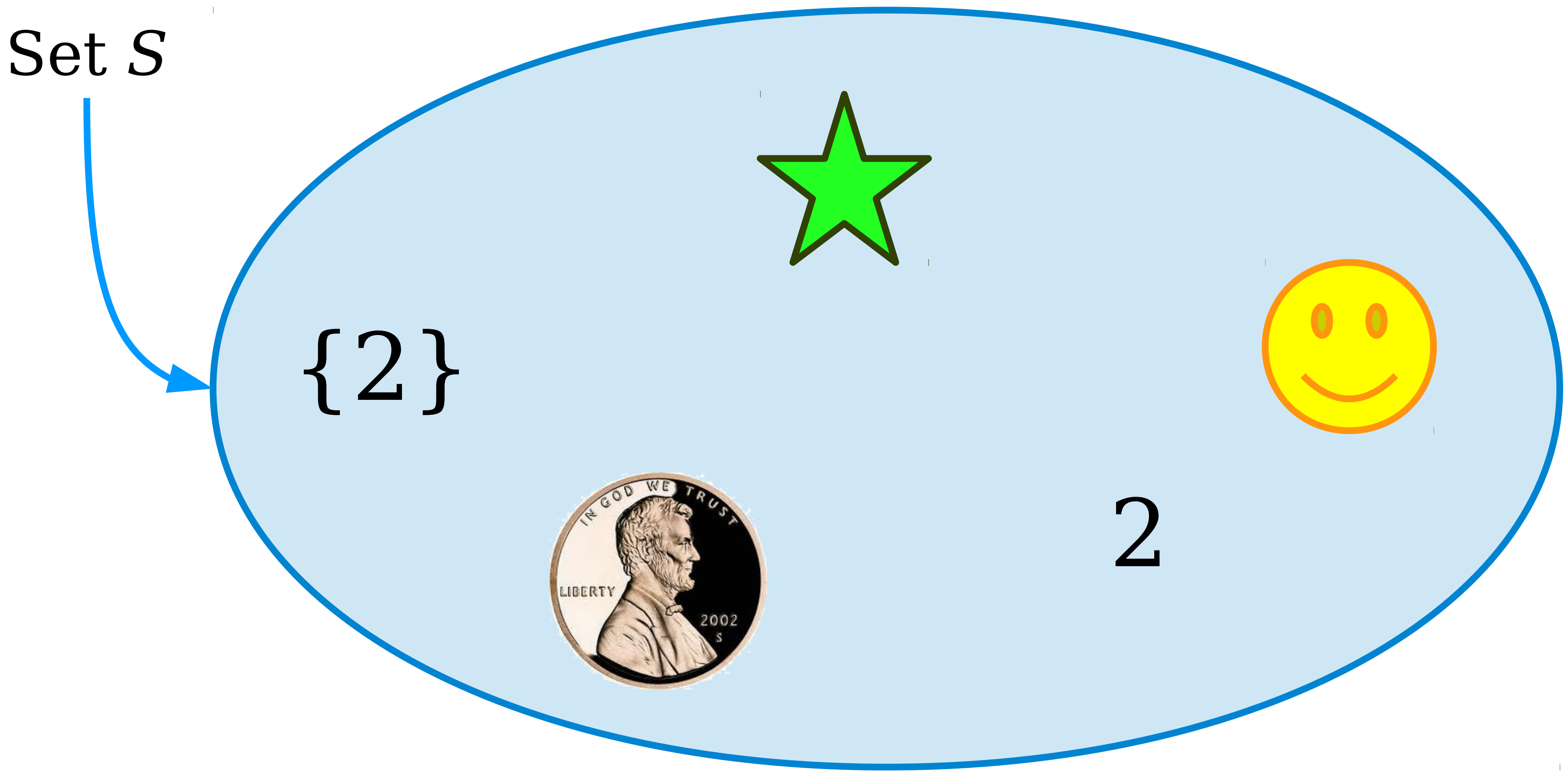
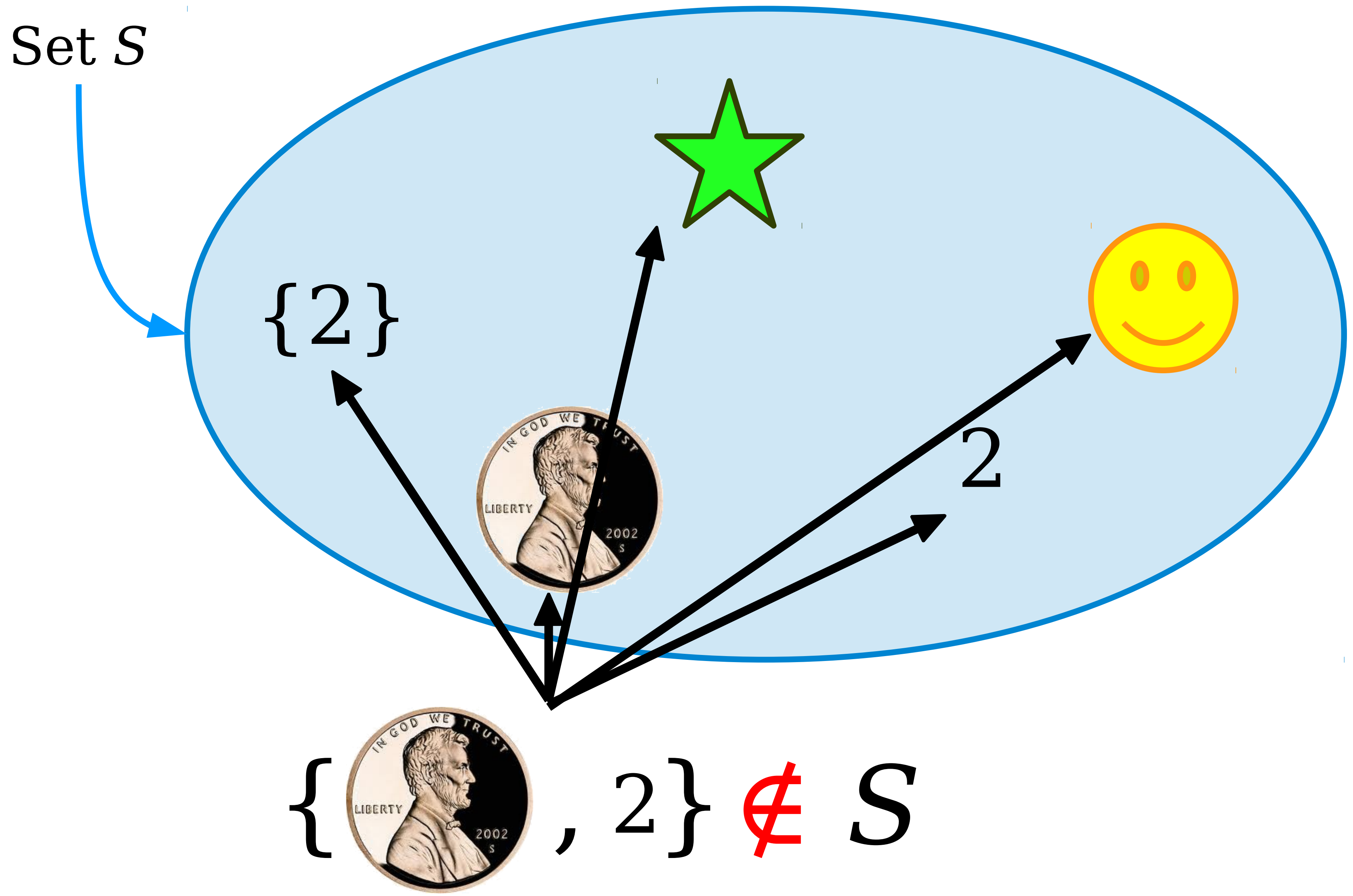


$\{$  $, 2\} \notin S$

Subsets and Elements



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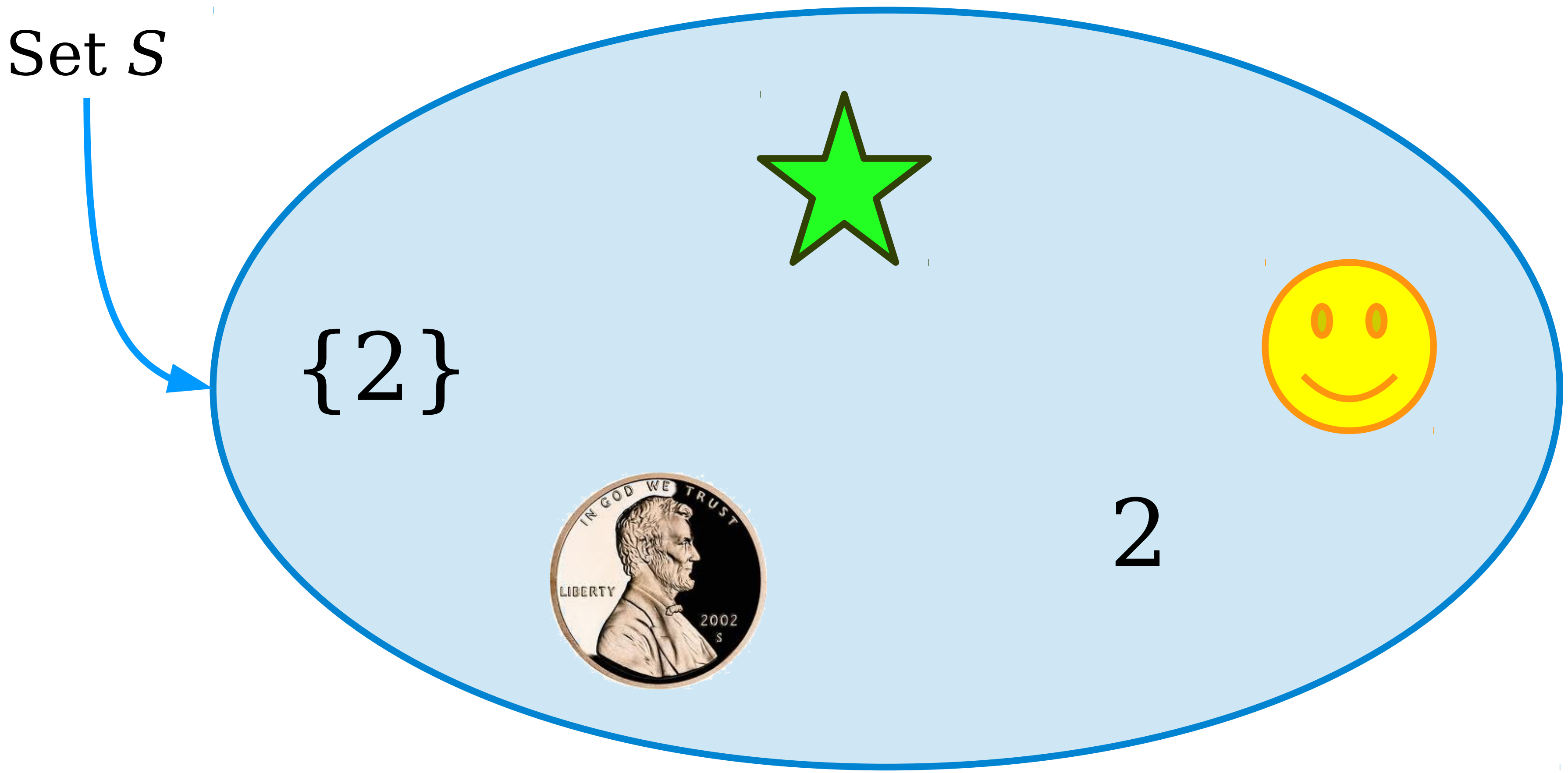

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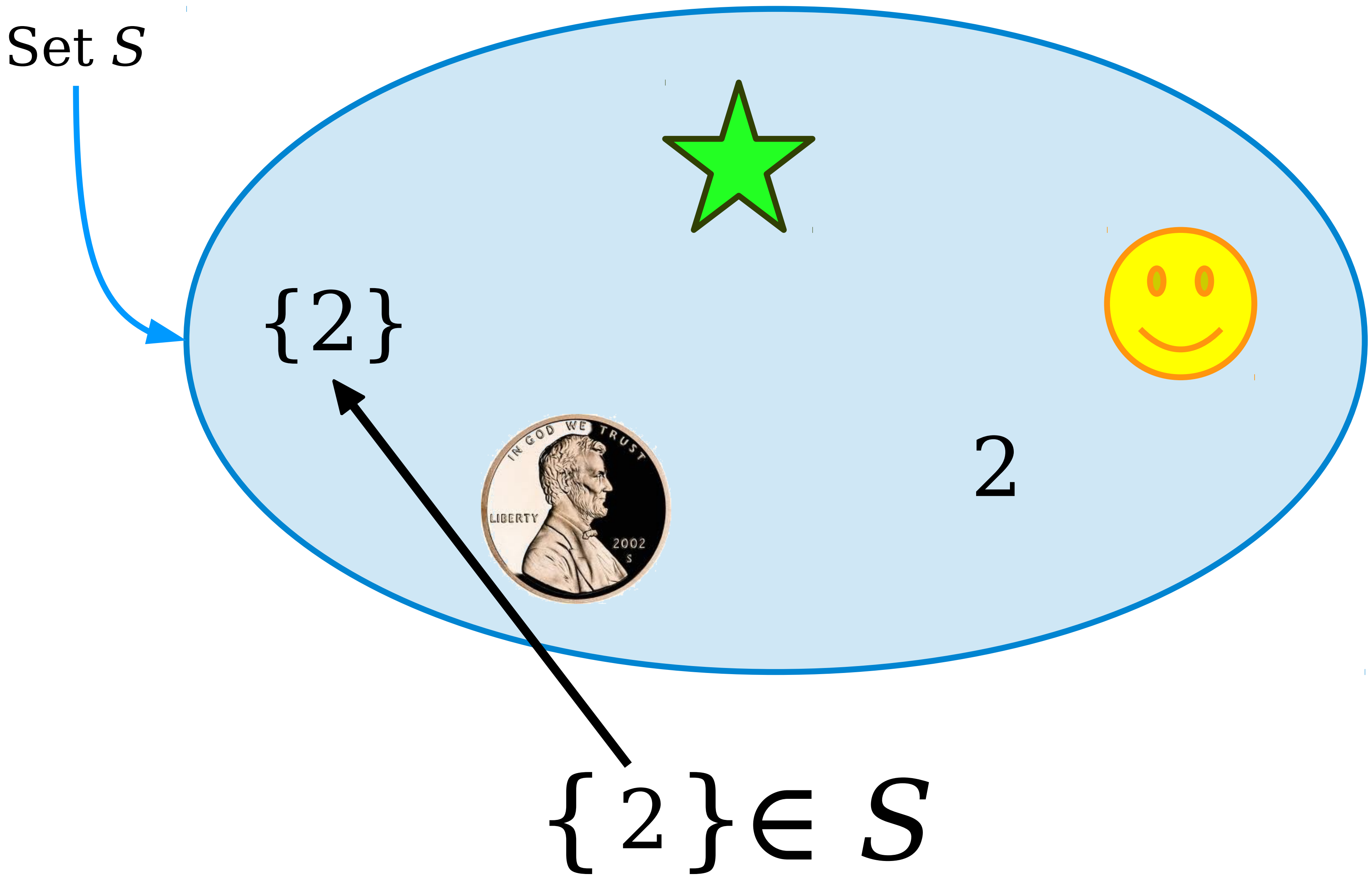

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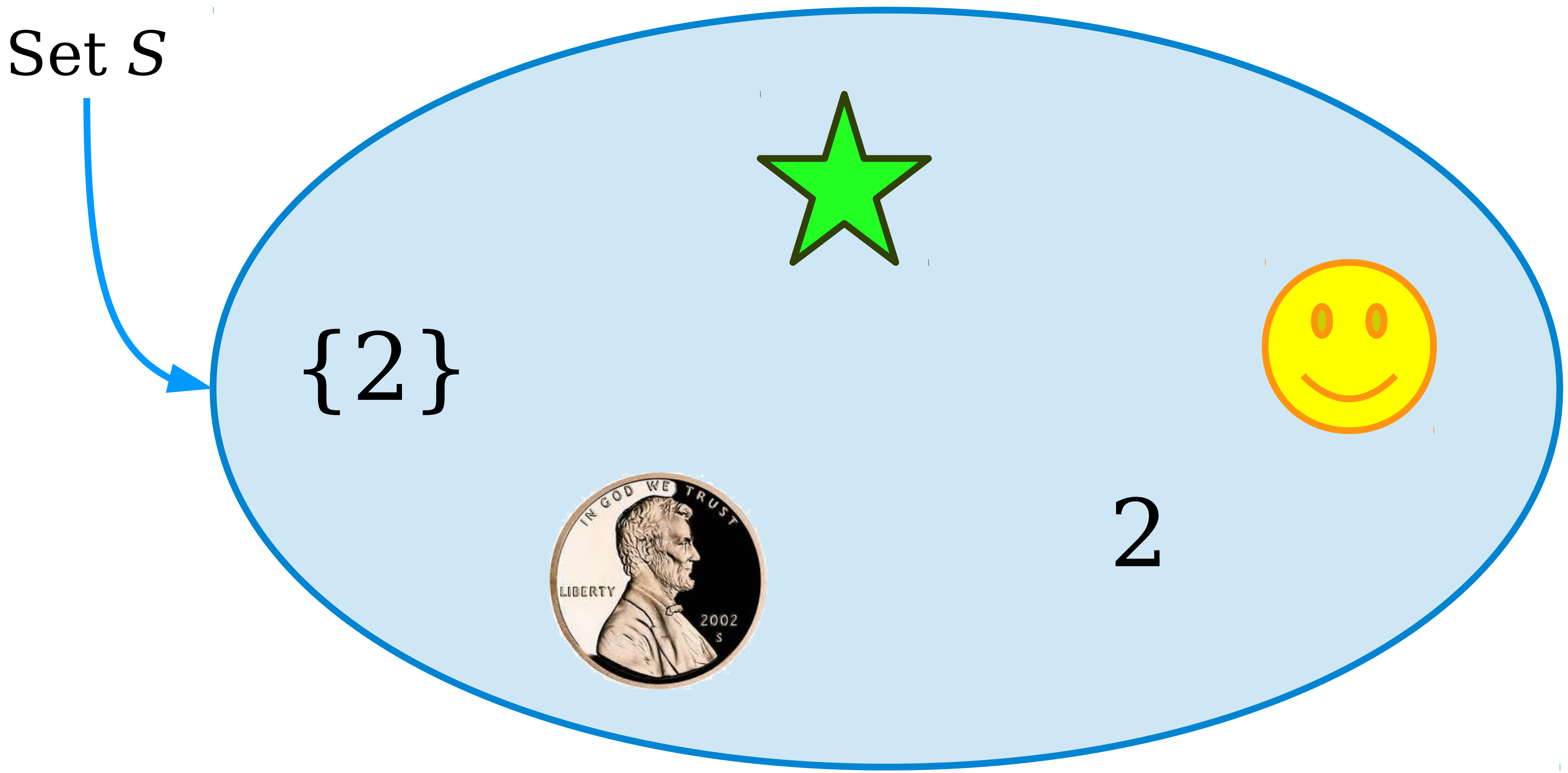

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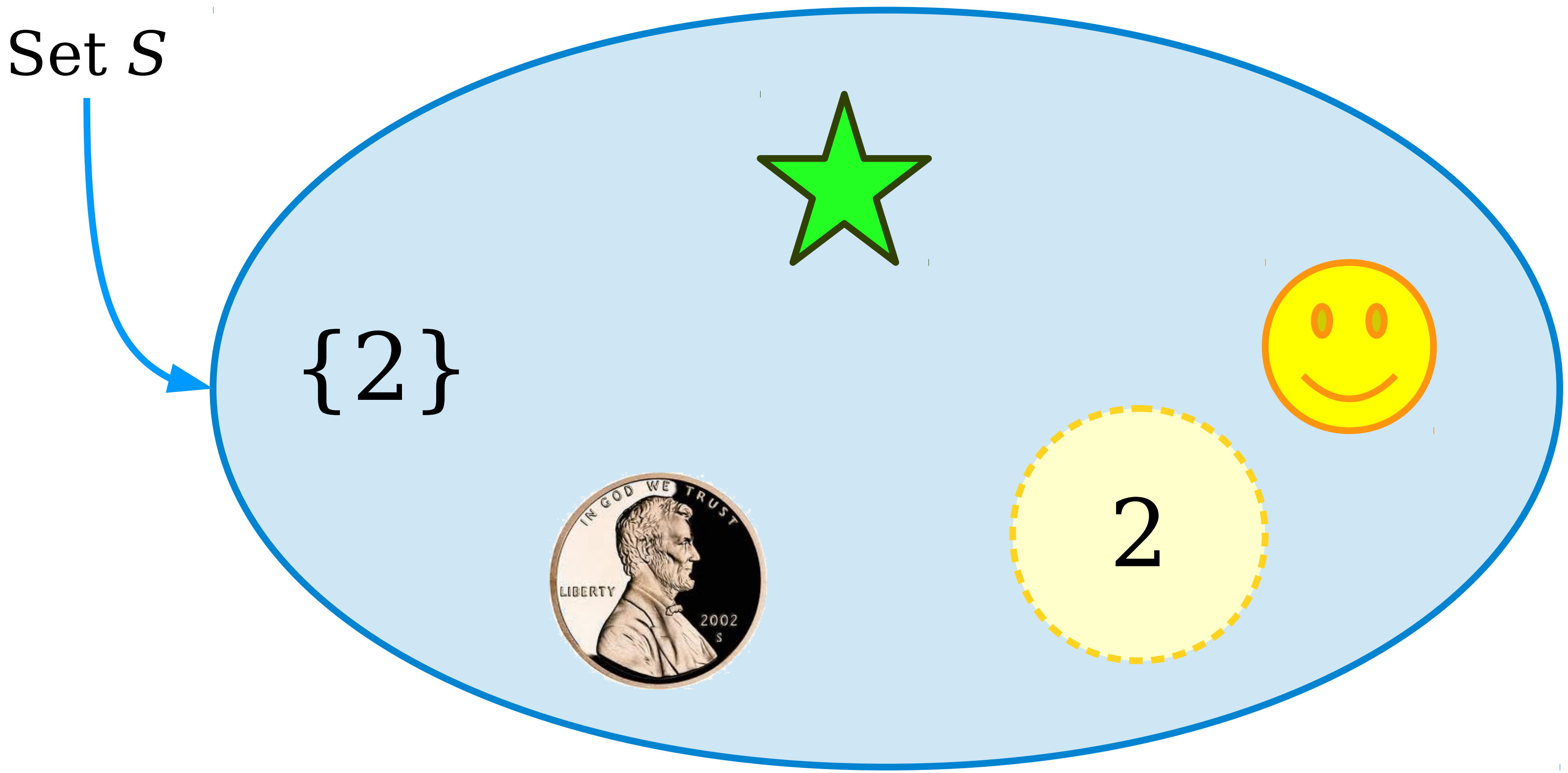

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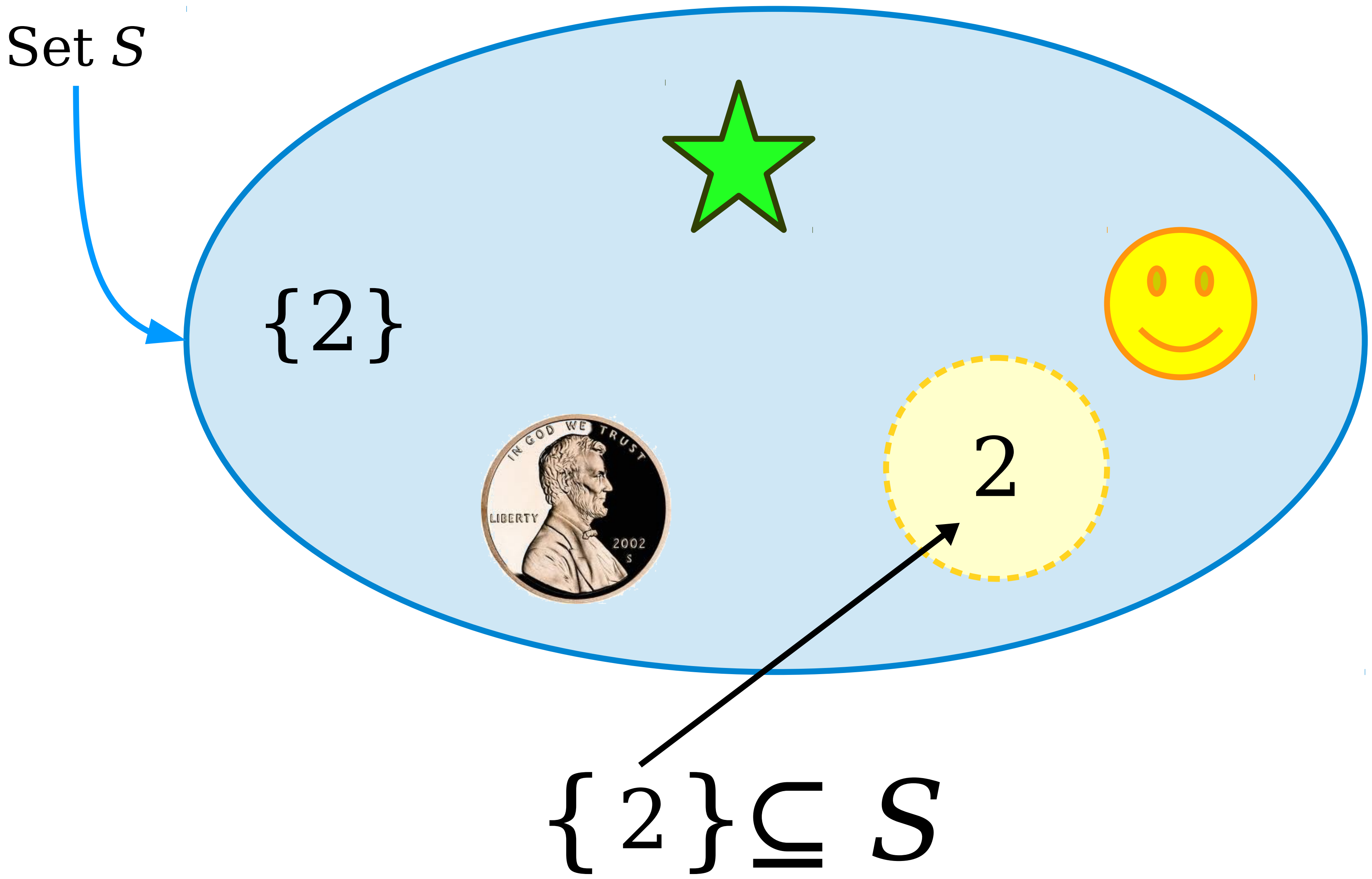

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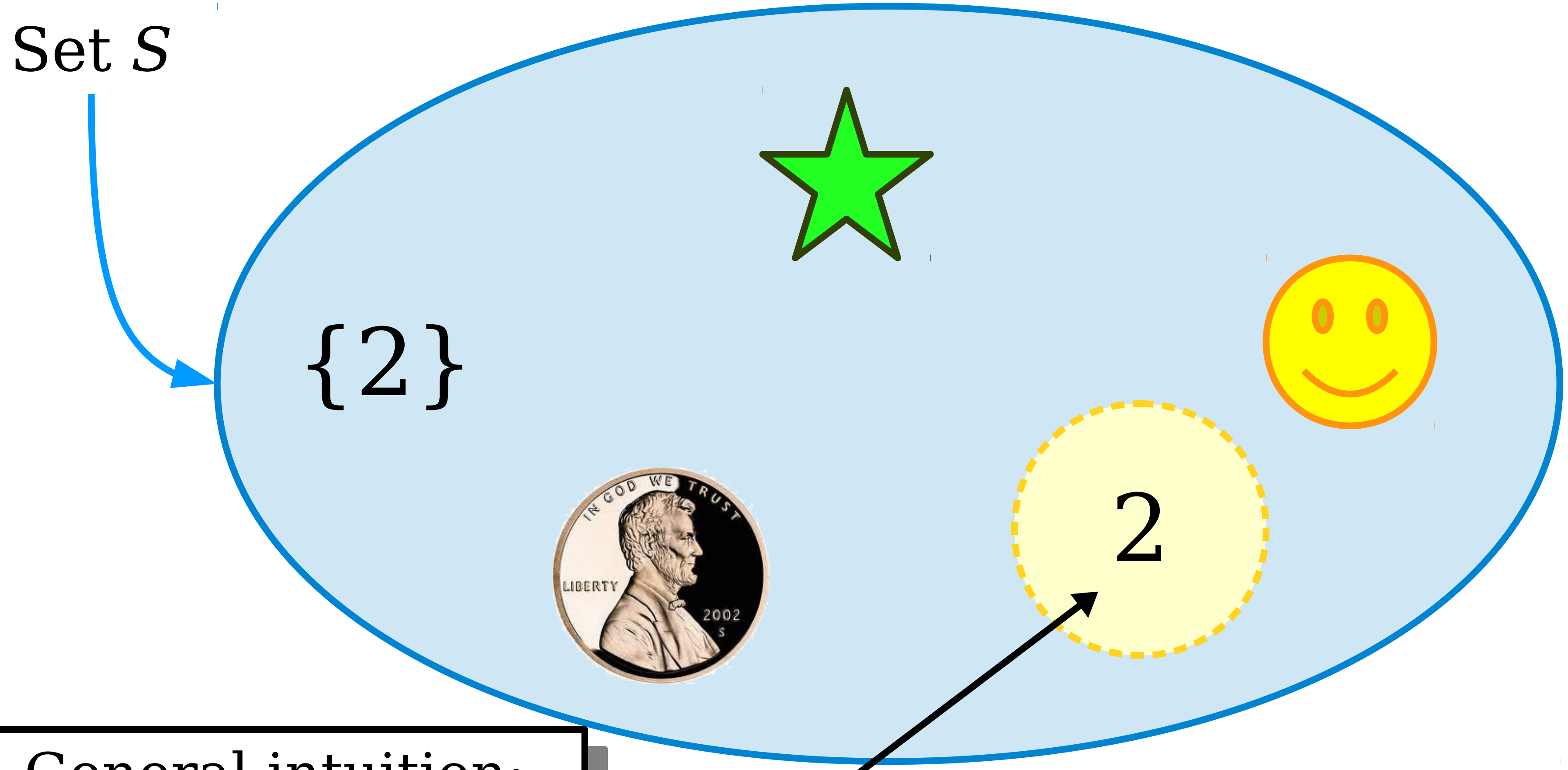
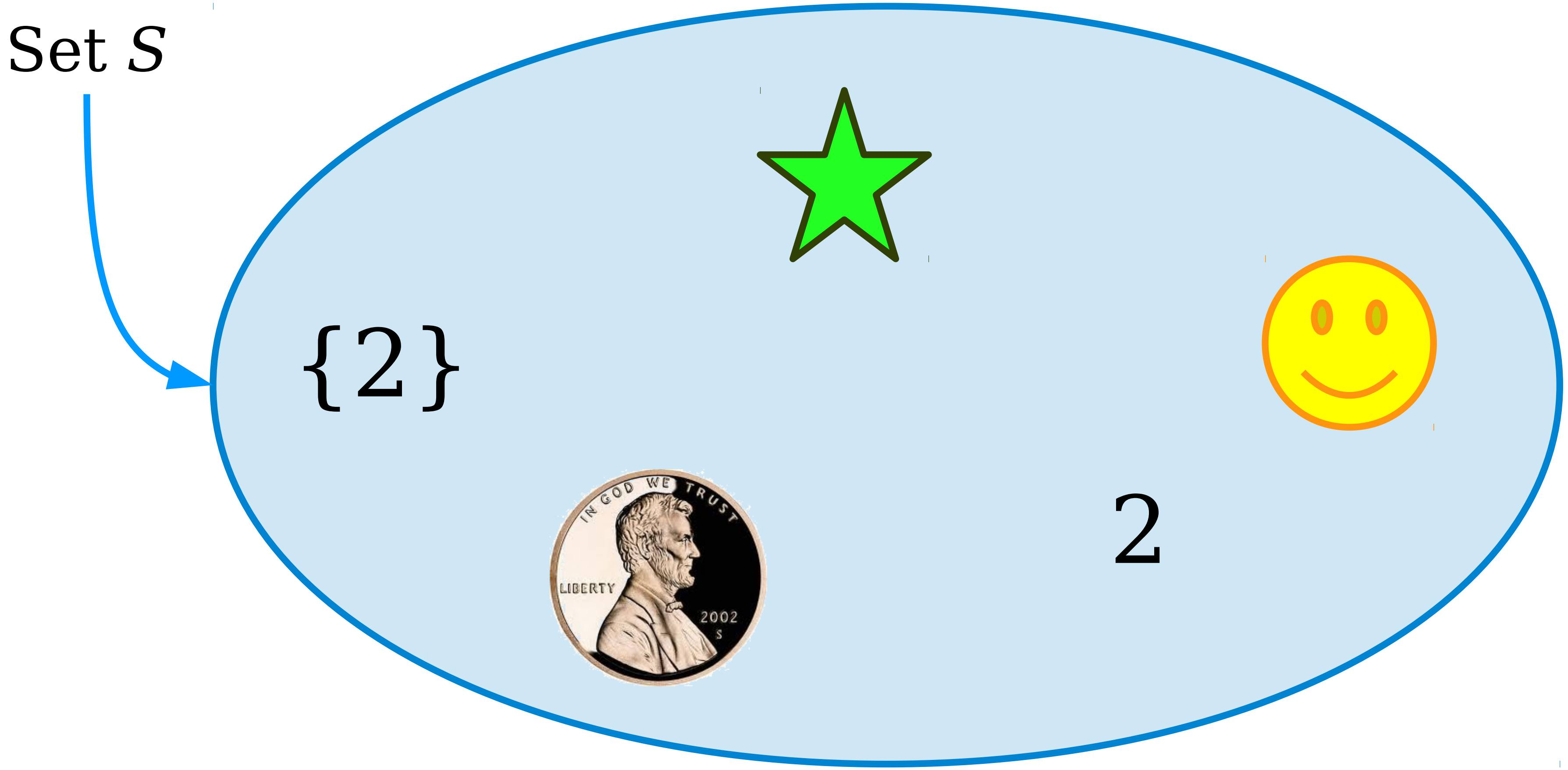


General intuition:
$A \subseteq B$ if you can form A by ***circling element(s) of B***.

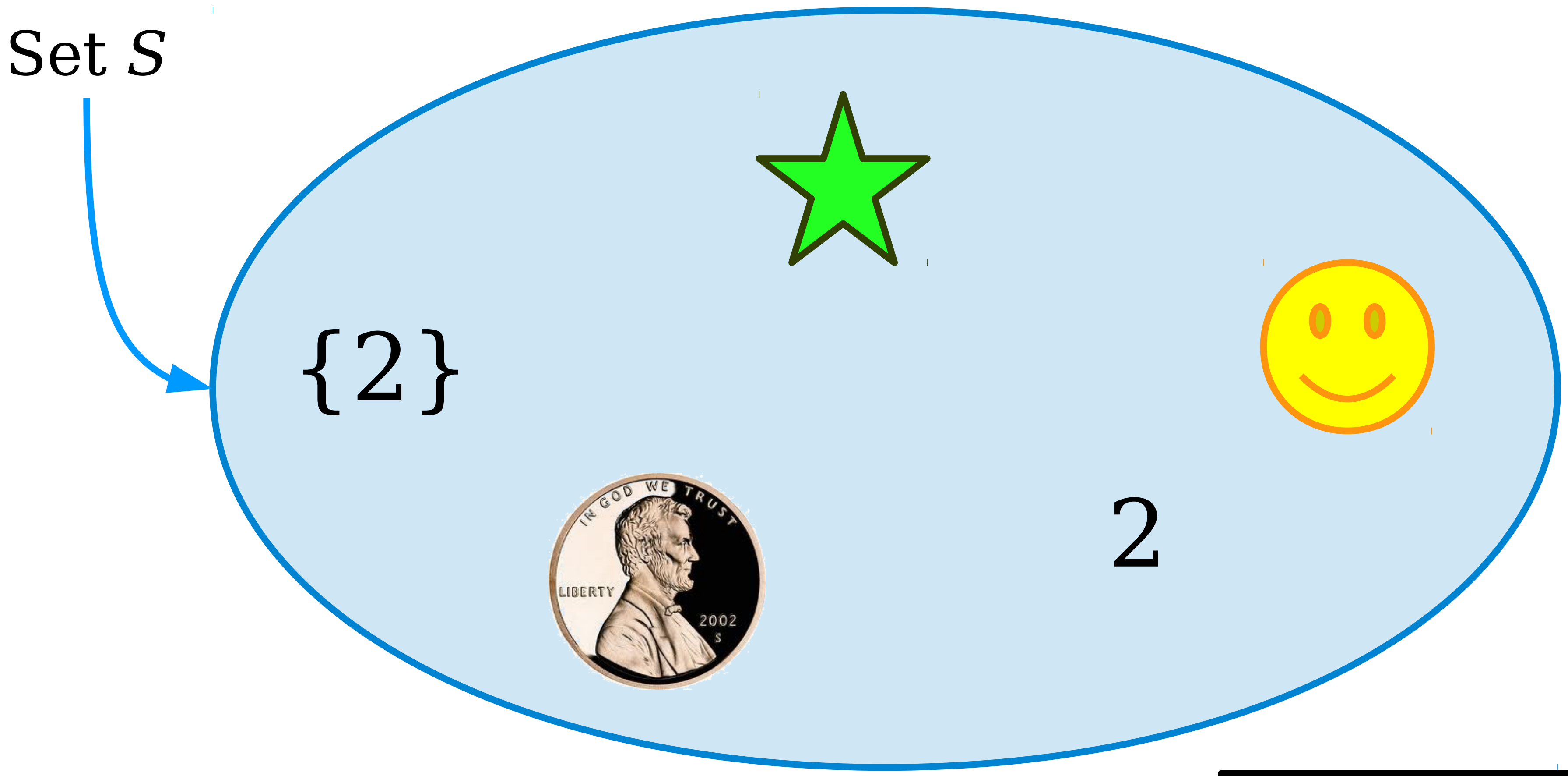
$\{2\} \subseteq S$

Subsets and Elements



$$2 \not\subseteq S$$

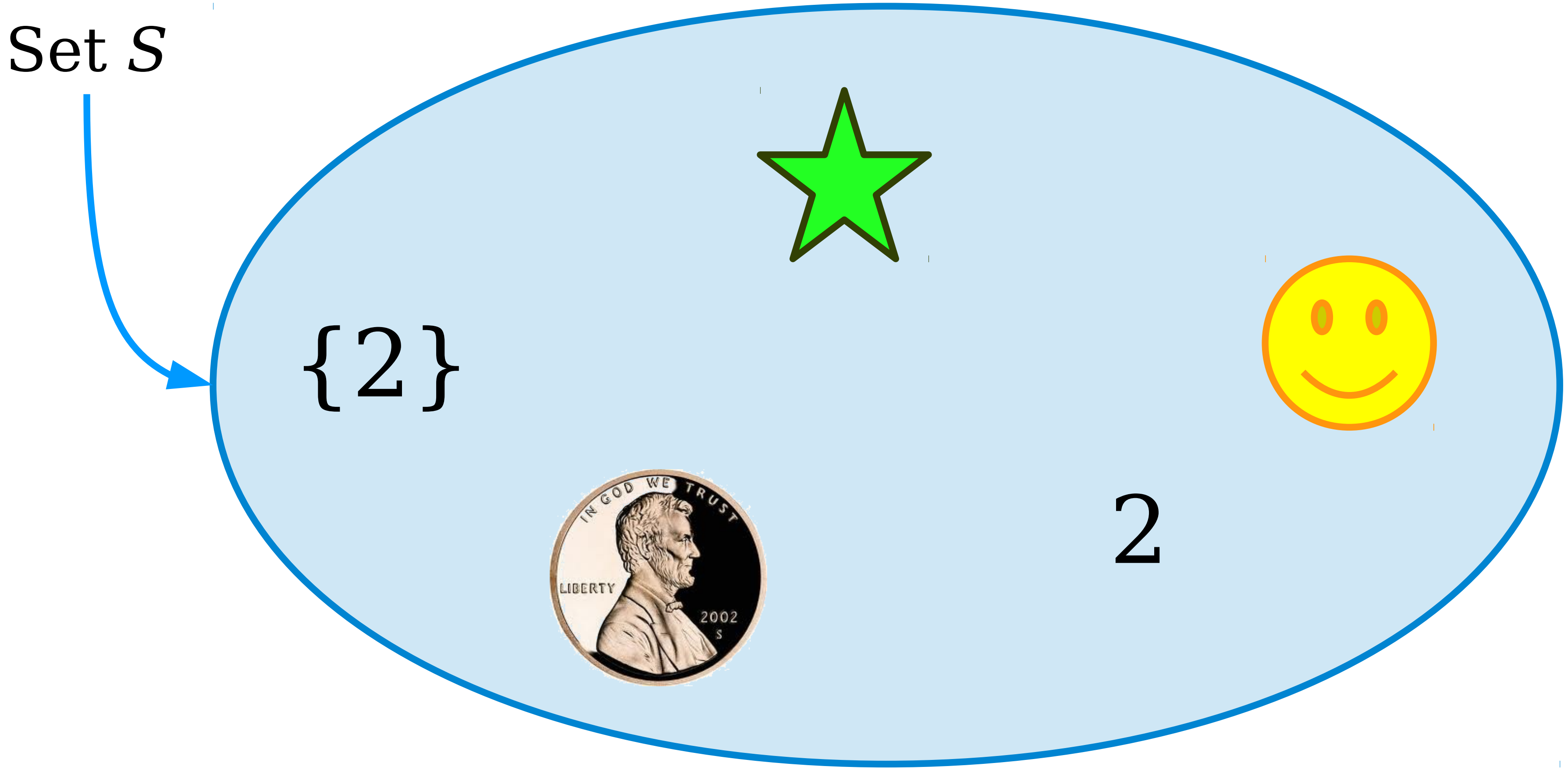
Subsets and Elements



$$2 \nsubseteq S$$

(Since 2 isn't a set.)

Subsets and Elements



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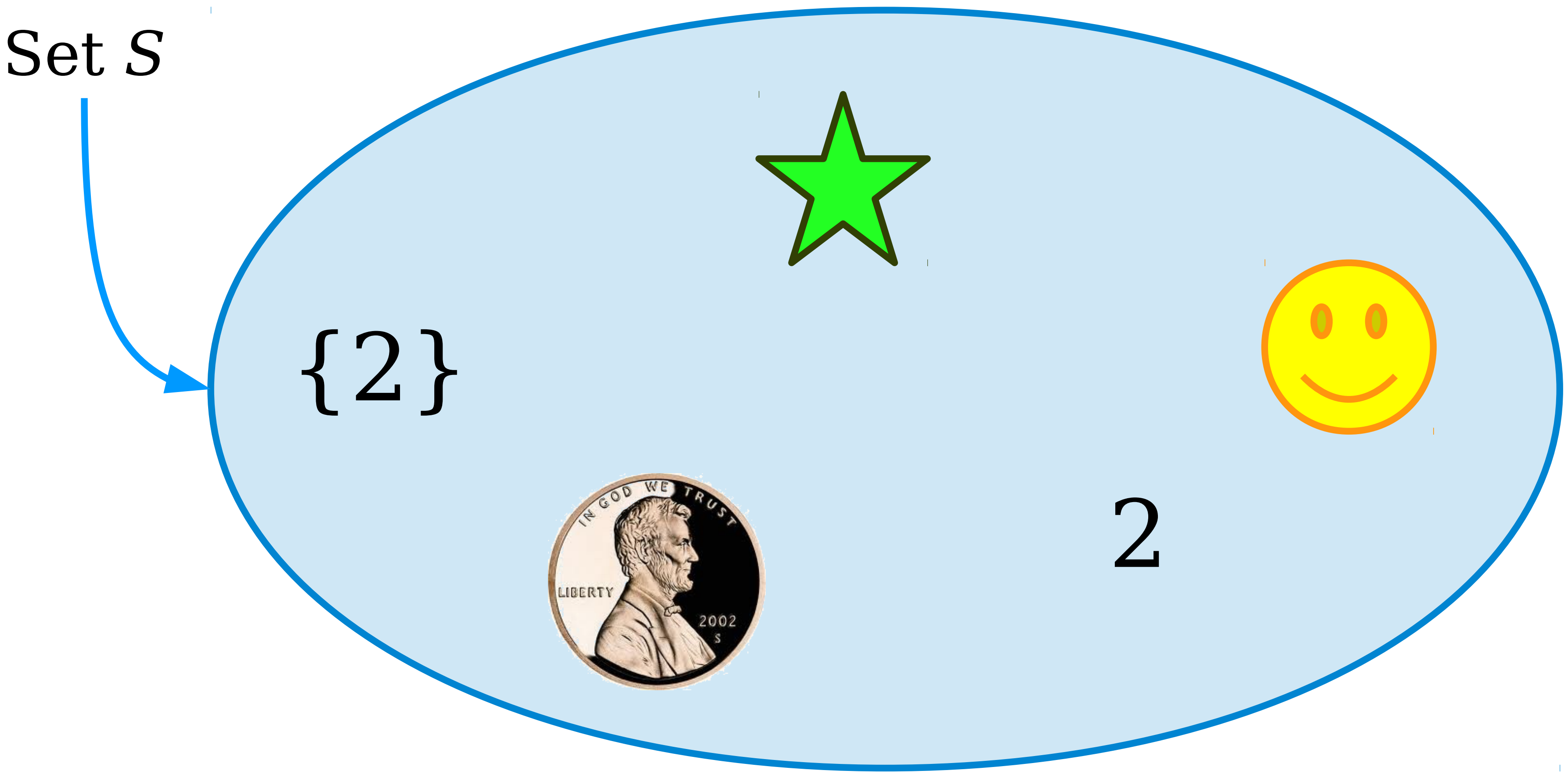
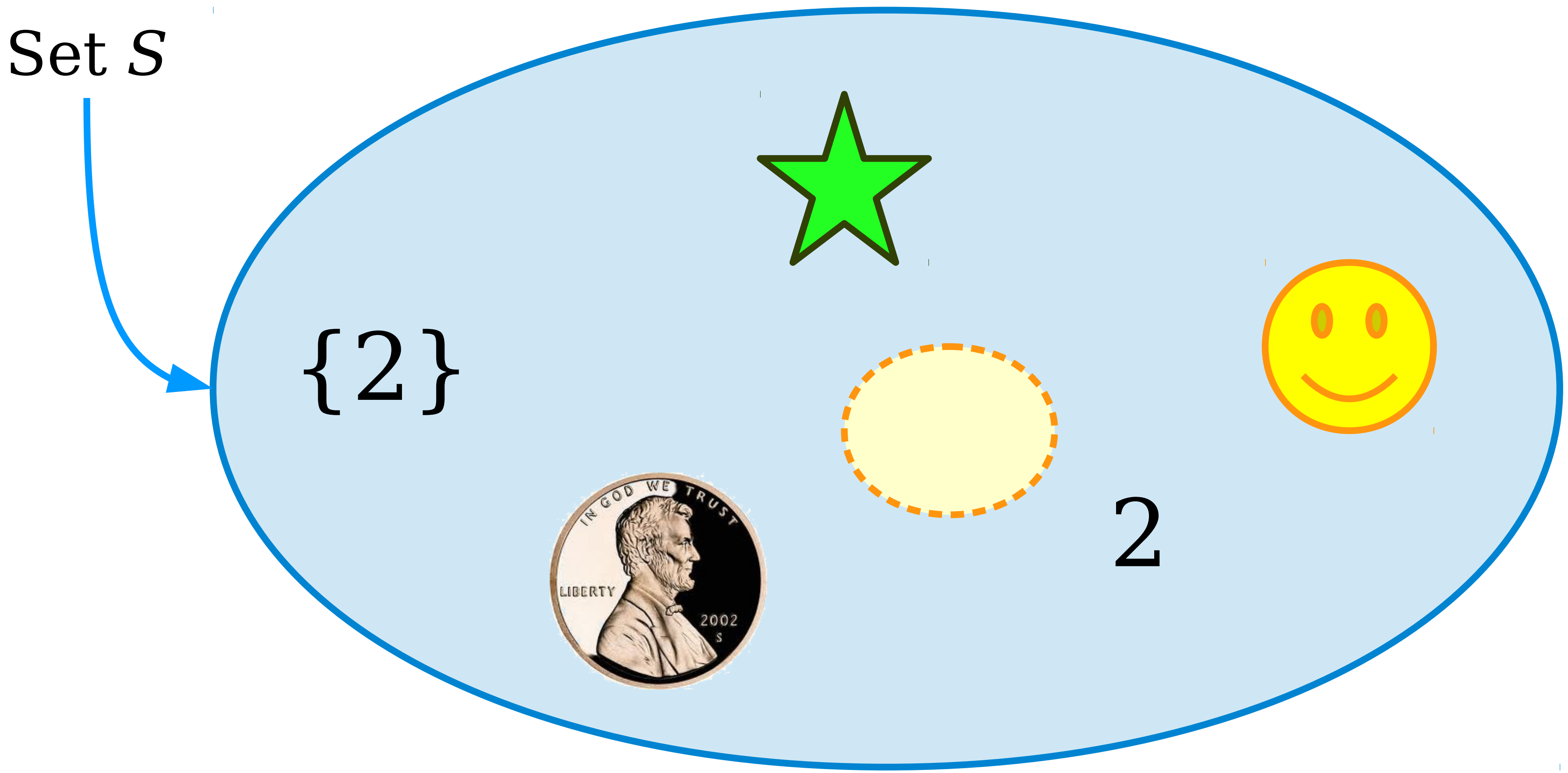


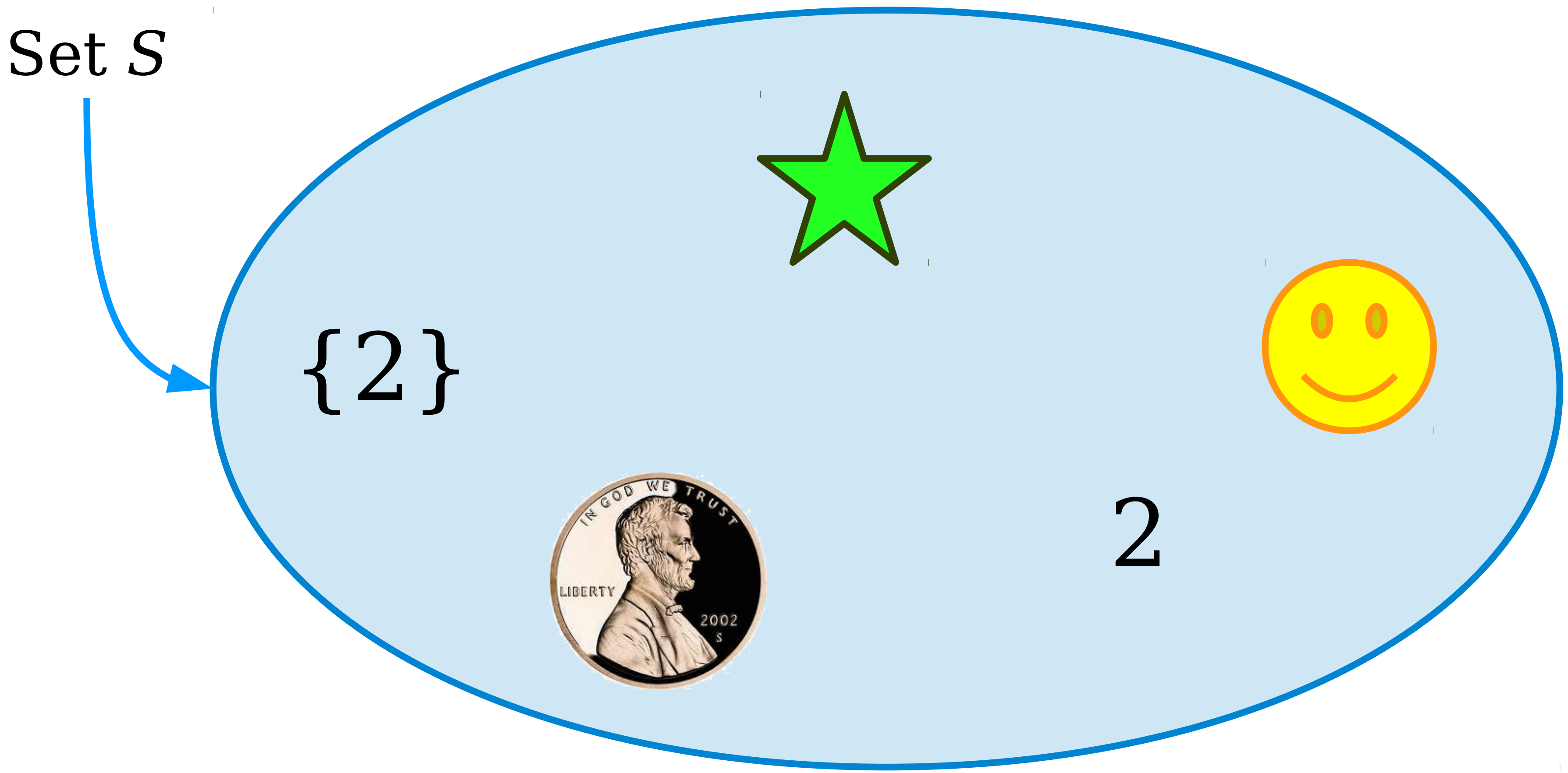
$$\varnothing \subseteq S$$

Subsets and Elements



$$\emptyset \subseteq S$$

Subsets and Elements



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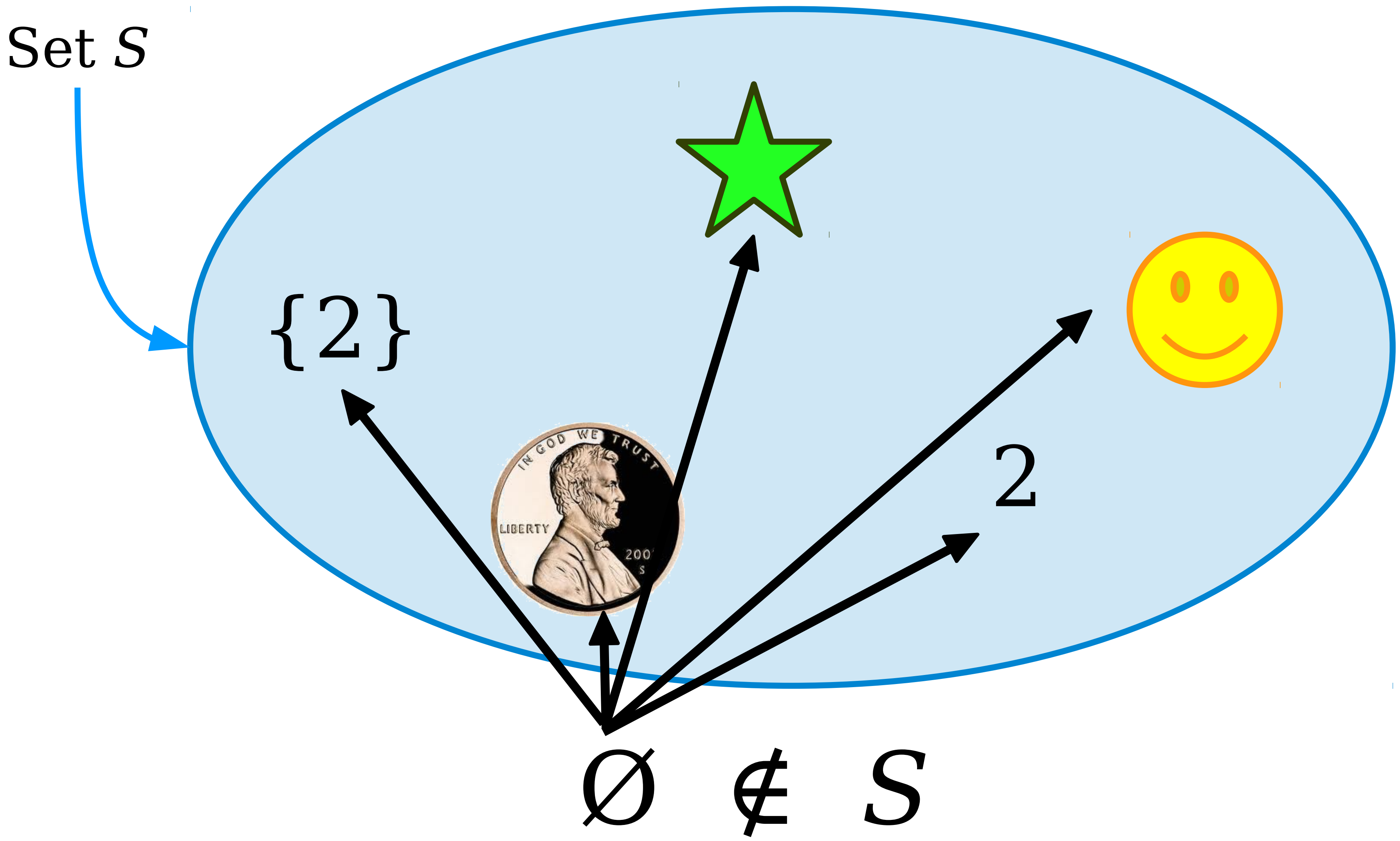
Subsets and Elements



Subsets and Elements

- We say that $\boldsymbol{S \in T}$ if, among the elements of T, one of them is *exactly* the object S.
- We say that $\boldsymbol{S \subseteq T}$ if S is a set and every element of S is also an element of T. (S has to be a set for the statement $S \subseteq T$ to be true.)
  - Although these concepts are similar, ***they are not the same!*** Not all elements of a set are subsets of that set and vice-versa.
  - We have a resource on the course website, the Guide to Elements and Subsets, that explores this in more depth.

$S = \{$  ,  $\}$

$\wp(S) = \{ \emptyset, \{$  $\}, \{$  $\}, \{$  ,  $\} \}$

This is the ***power set*** of S, the set of all subsets of S. We write the power set of S as $\wp(S)$.

Formally, $\wp(S) = \{ T \mid T \subseteq S \}$.

(Do you see why?)

What is $\wp(\emptyset)$?

Answer: $\{\emptyset\}$

Remember that $\emptyset \neq \{\emptyset\}$!

Cardinality

Cardinality

Cardinality

- The ***cardinality*** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality as $|S|$.
- Examples:
  - $|\{whimsy, mirth\}| = 2$
  - $|\{\{a, b\}, \{c, d, e, f, g\}, \{h\}\}| = 3$
  - $|\{1, 2, 3, 3, 3, 3, 3\}| = 3$
  - $|\{ n \in \mathbb{N} \mid n < 4 \}| = |\{0, 1, 2, 3\}| = 4$
  - $| \emptyset | = 0$
  - $| \{\emptyset\} | = 1$

The Cardinality of $\mathbb{N}$

- What is $|\mathbb{N}|$?
  - There are infinitely many natural numbers.
  - $|\mathbb{N}|$ can't be a natural number, since it's infinitely large.

The Cardinality of ℕ

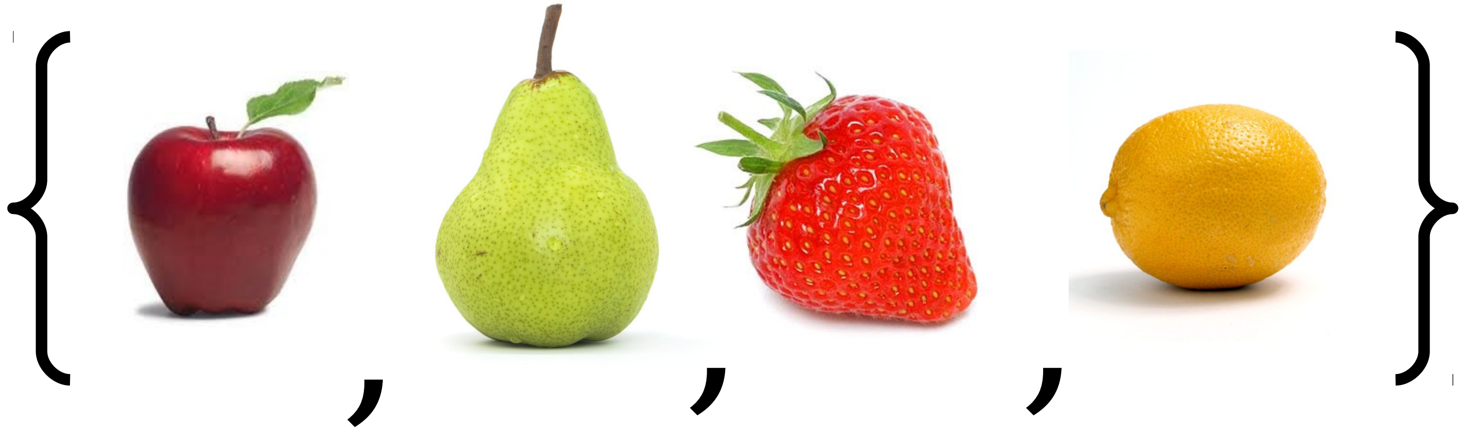
- What is $|\mathbb{N}|$?
  - There are infinitely many natural numbers.
  - $|\mathbb{N}|$ can't be a natural number, since it's infinitely large.
- We need to introduce a new term.
- Let's define $\aleph_0 = |\mathbb{N}|$.
  - $\aleph_0$ is pronounced “aleph-zero,” “aleph-nought,” or “aleph-null.”

Consider the set

$$S = \{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}.$$

What is $|S|$?

How Big Are These Sets?



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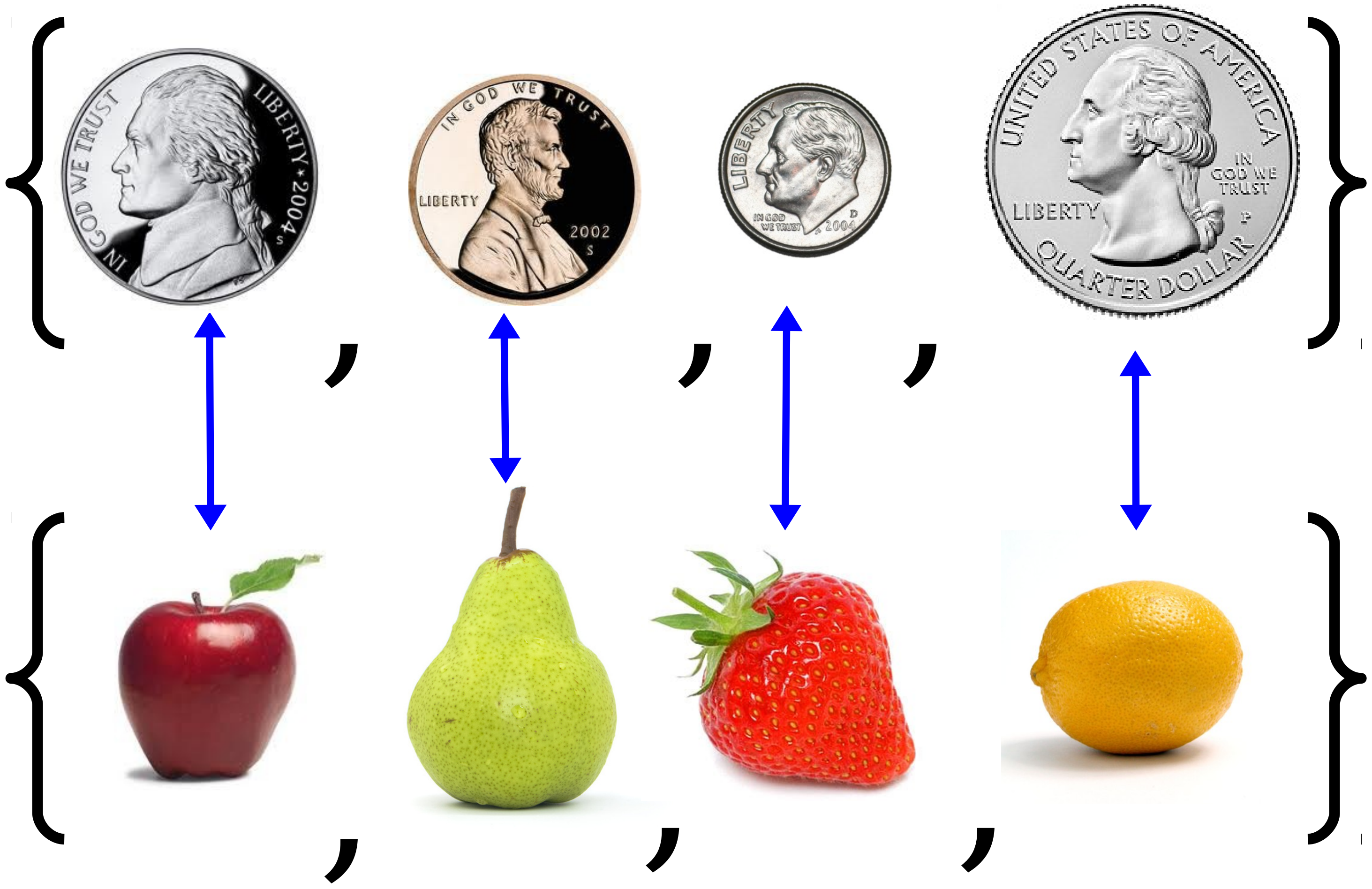
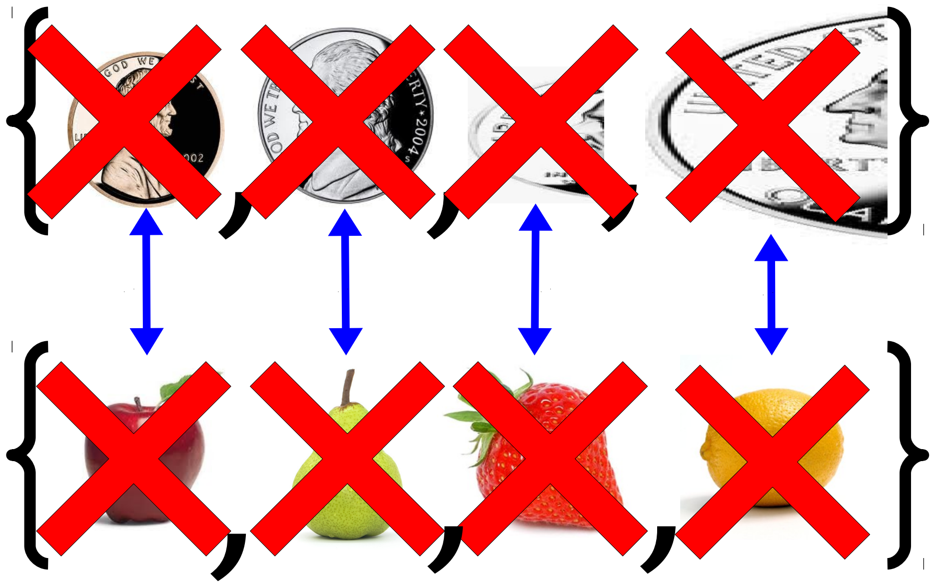


Comparing Cardinalities

- *By definition*, two sets have the same size if there is a way to pair their elements off without leaving any elements uncovered.
- The intuition:



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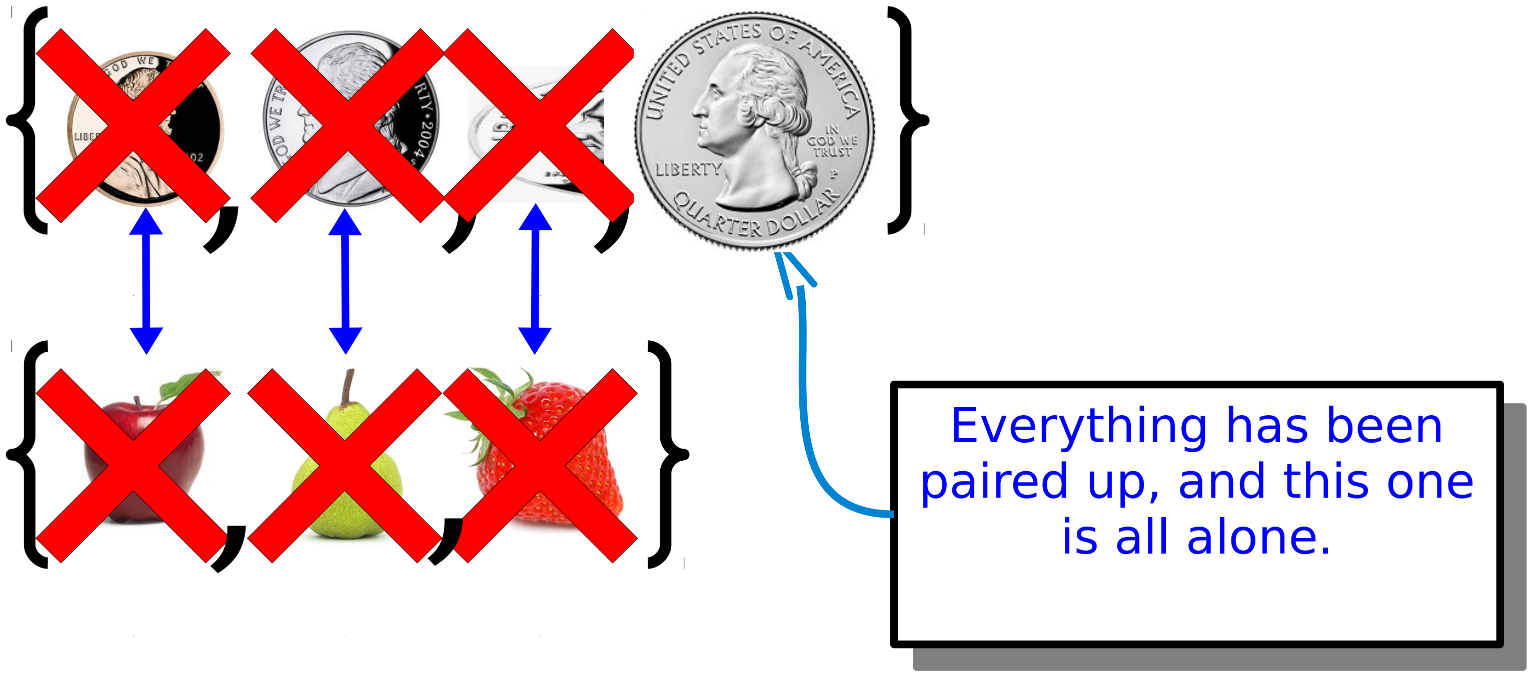


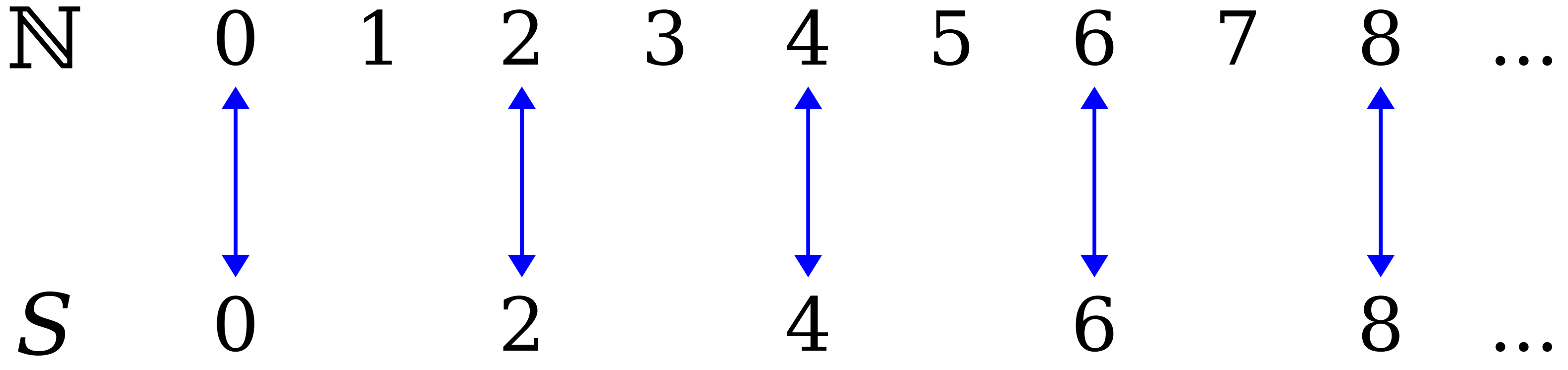
Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	…
	↕		↕		↕		↕		↕	
S	0		2		4		6		8	…

$S = \{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$

Two sets have the same size if there is a way to pair their elements off without leaving any elements uncovered

Infinite Cardinalities



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Two sets have the same size if *there is a way* to pair their elements off without leaving any elements uncovered

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	…
S	0		2		4		6		8	…

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Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	…
	↕	↕	↕	↕	↕	↕	↕	↕	↕	
S	0	2	4	6	8	10	12	14	16	…

$$n \leftrightarrow 2n$$

$$S = \{ \ n \mid n \in \mathbb{N} \text{ and } n \text{ is even } \}$$

$$|S| = |\mathbb{N}| = \aleph_0$$

Important Question:

Do all infinite sets have the same cardinality?

$S = \{$ 

, 

$\}$

$\wp(S) = \{ \varnothing, \{$ 

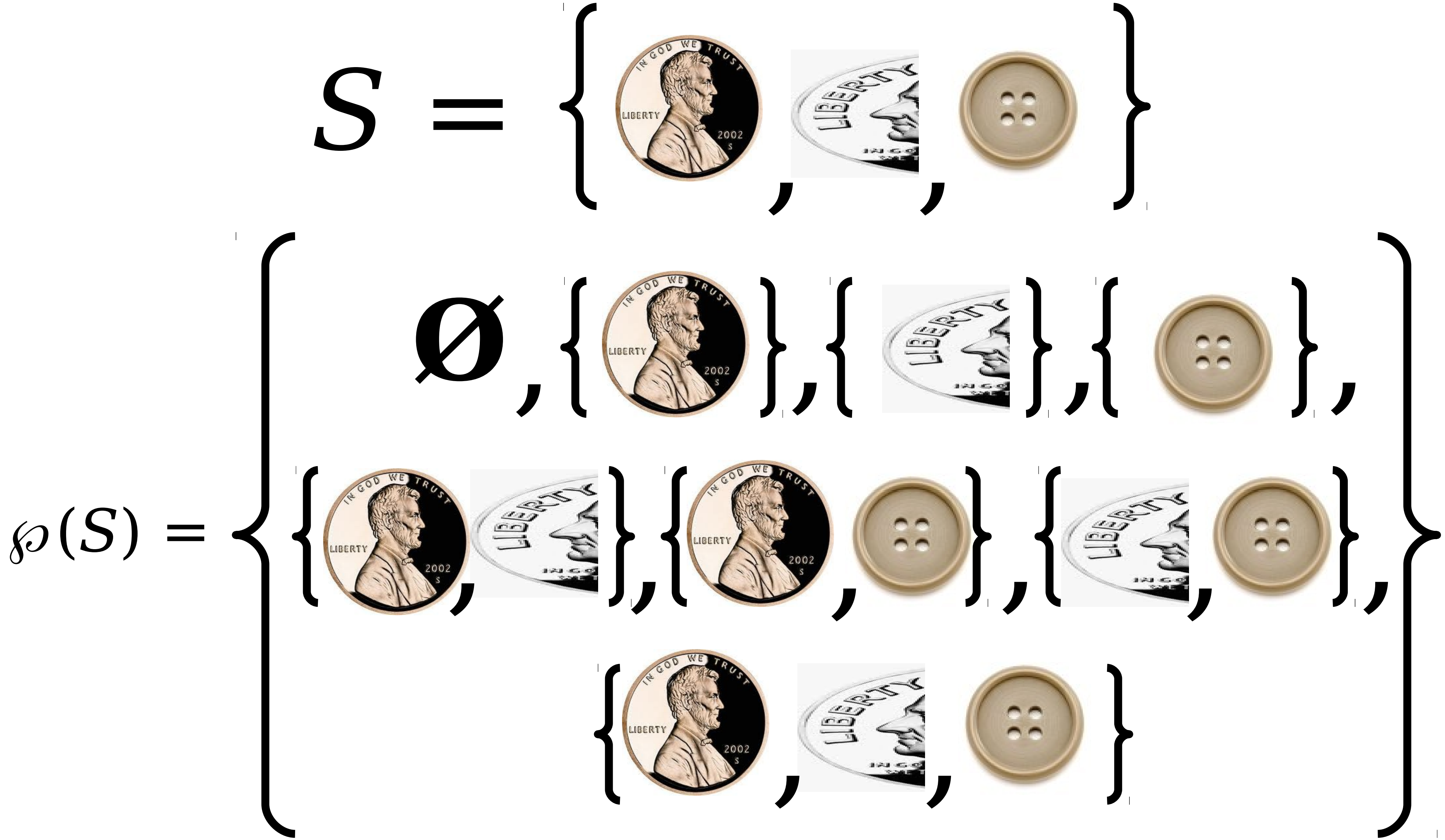
$\}, \{$ 

$\}, \{$ 

, 

$\} \}$

$$\mathbf{|S| < |\wp(S)|}$$



$$|S| < |\wp(S)|$$

$$S = \{a, b, c, d\}$$

$$\wp(S) = \{$$

$$\varnothing,$$

$$\{a\}, \{b\}, \{c\}, \{d\},$$

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$$

$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\},$$

$$\{a, b, c, d\}$$

$$\}$$

$$|S| < |\wp(S)|$$

If $|S|$ is infinite, what is the relation between $|S|$ and $|\wp(S)|$?

Does $|S| = |\wp(S)|$?

If $|S| = |\wp(S)|$, we can pair up the elements of S and the elements of $\wp(S)$ without leaving anything out.

If $|S| = |\wp(S)|$, we can pair up the elements of S and the elements of $\wp(S)$ without leaving anything out.

If $|S| = |\wp(S)|$, we can pair up the elements of S and the subsets of S without leaving anything out.

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What would that look like?

$$x_0 \longleftrightarrow \{\, x_0,\ x_2,\ x_4,\ \dots \,\}$$

$$x_1 \longleftrightarrow \{\, x_3,\ x_5,\ \dots \,\}$$

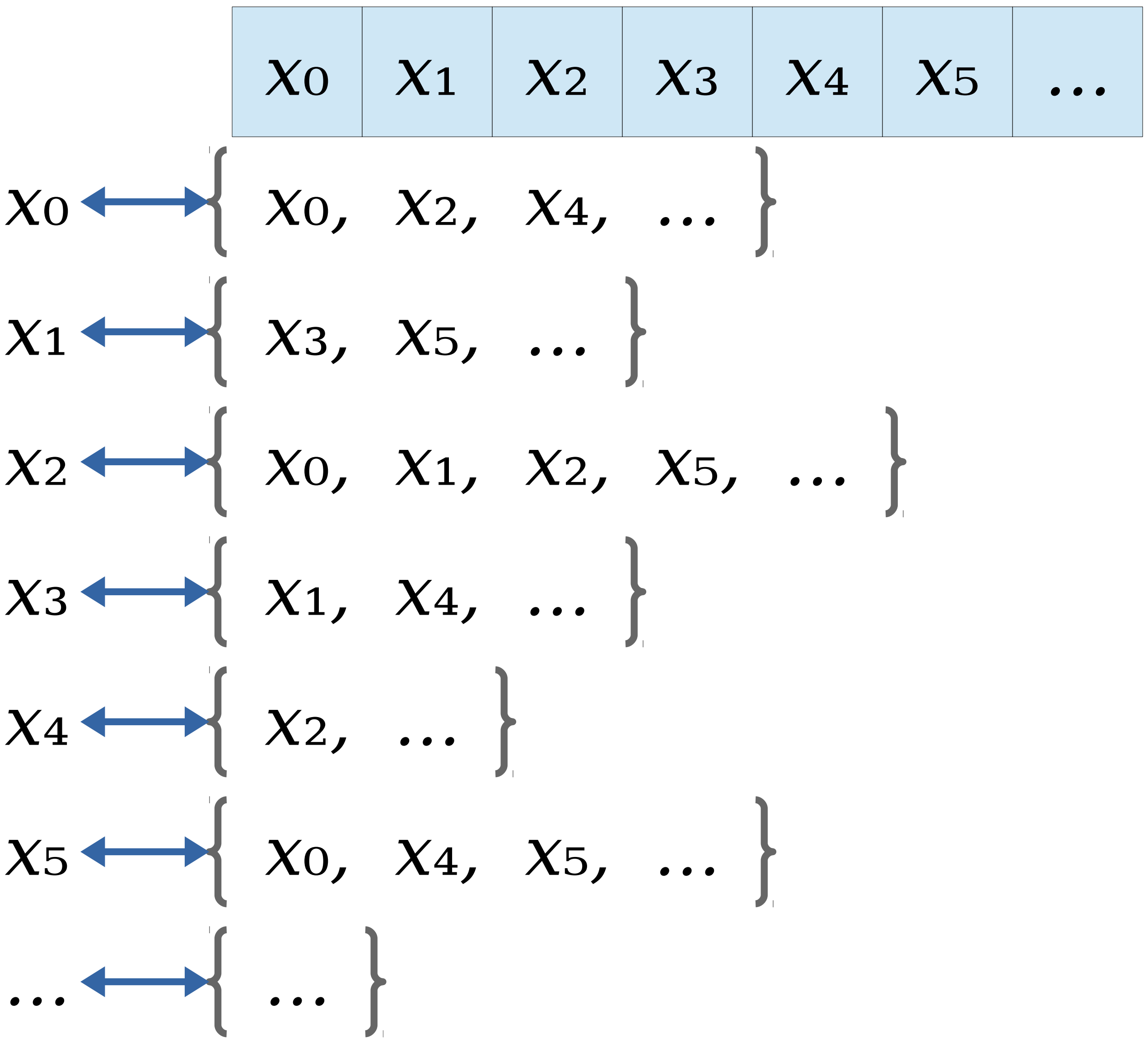
$$x_2 \longleftrightarrow \{\, x_0,\ x_1,\ x_2,\ x_5,\ \dots \,\}$$

$$x_3 \longleftrightarrow \{\, x_1,\ x_4,\ \dots \,\}$$

$$x_4 \longleftrightarrow \{\, x_2,\ \dots \,\}$$

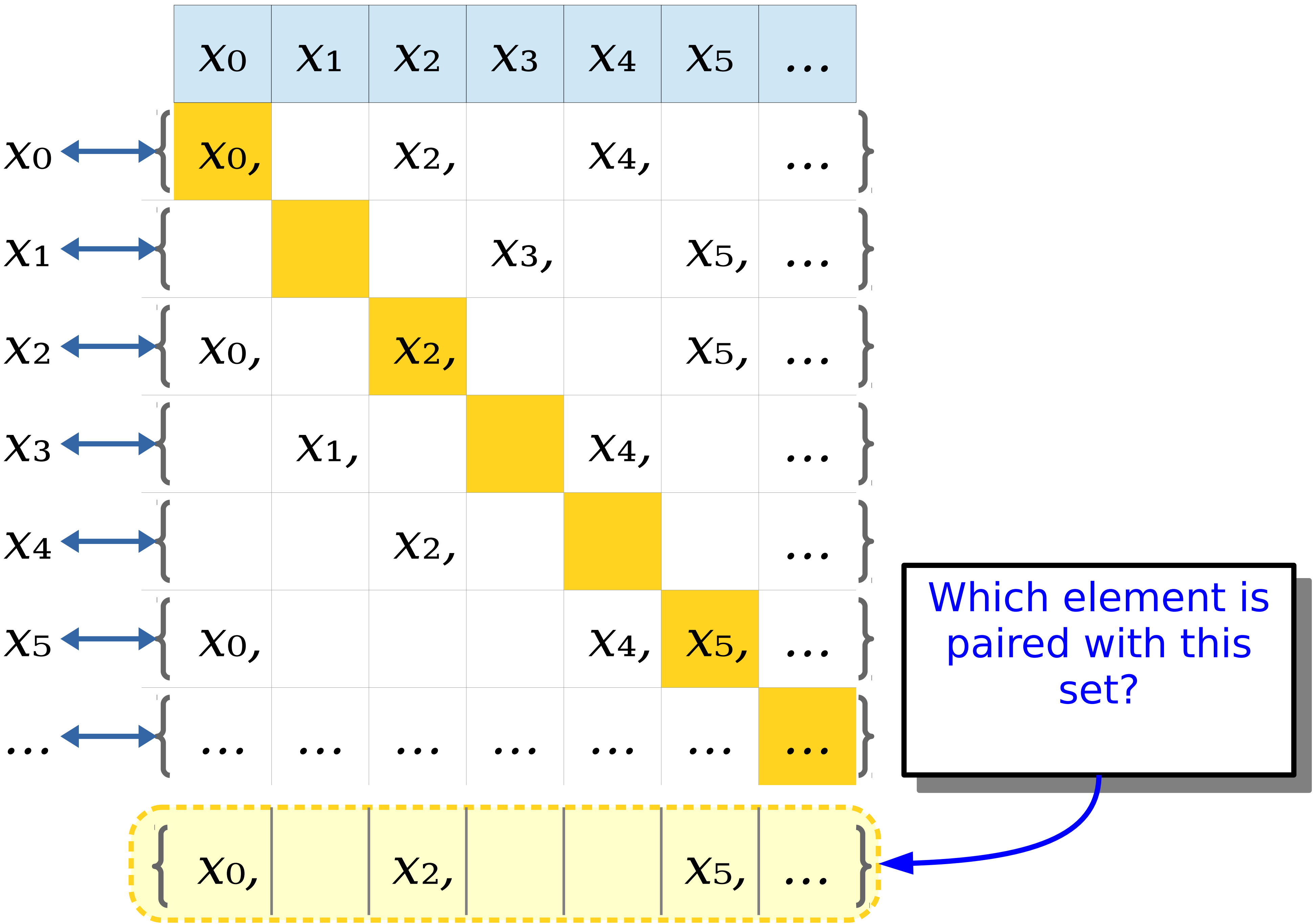
$$x_5 \longleftrightarrow \{\, x_0,\ x_4,\ x_5,\ \dots \,\}$$

$$\dots \longleftrightarrow \{\, \dots \,\}$$

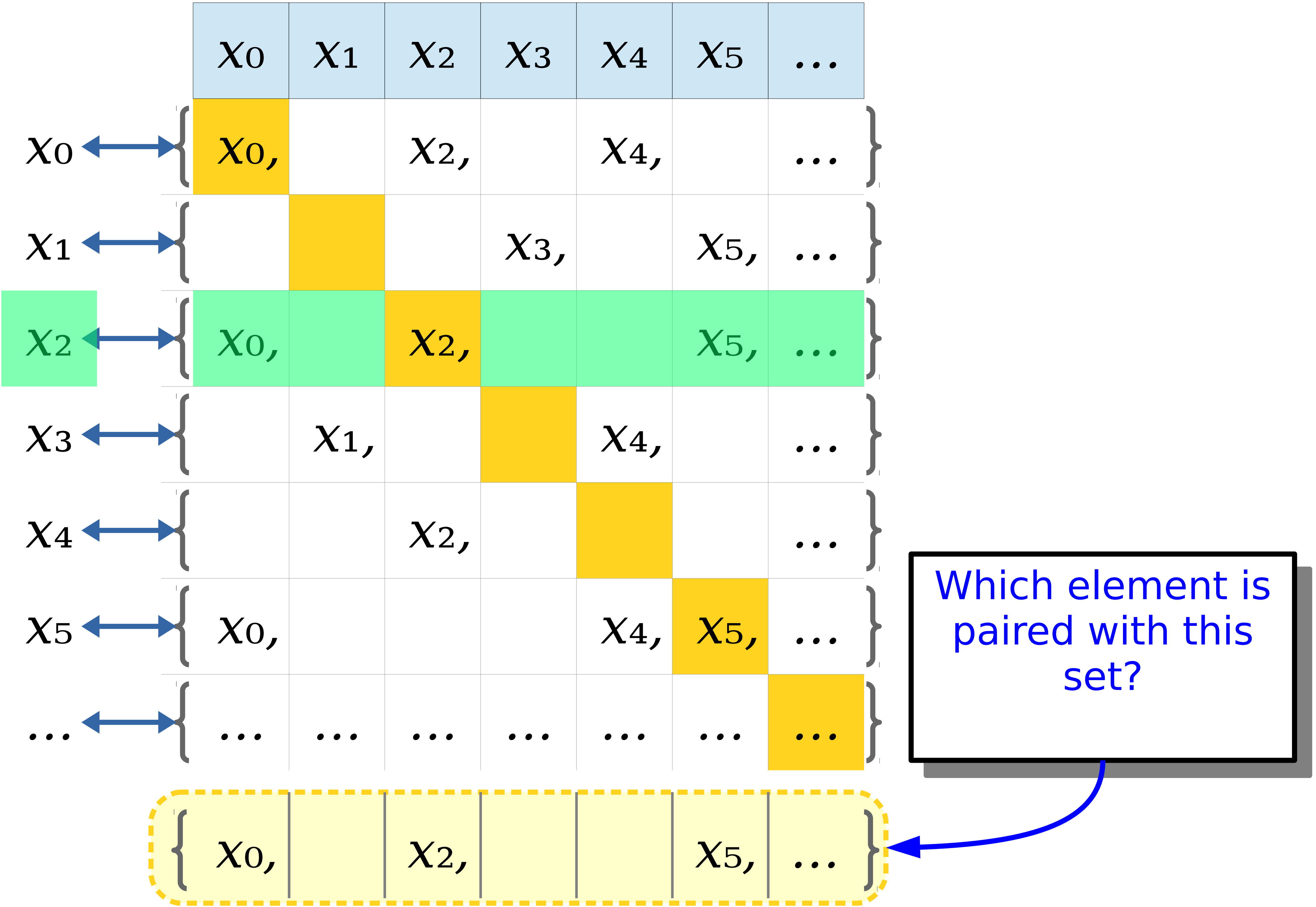


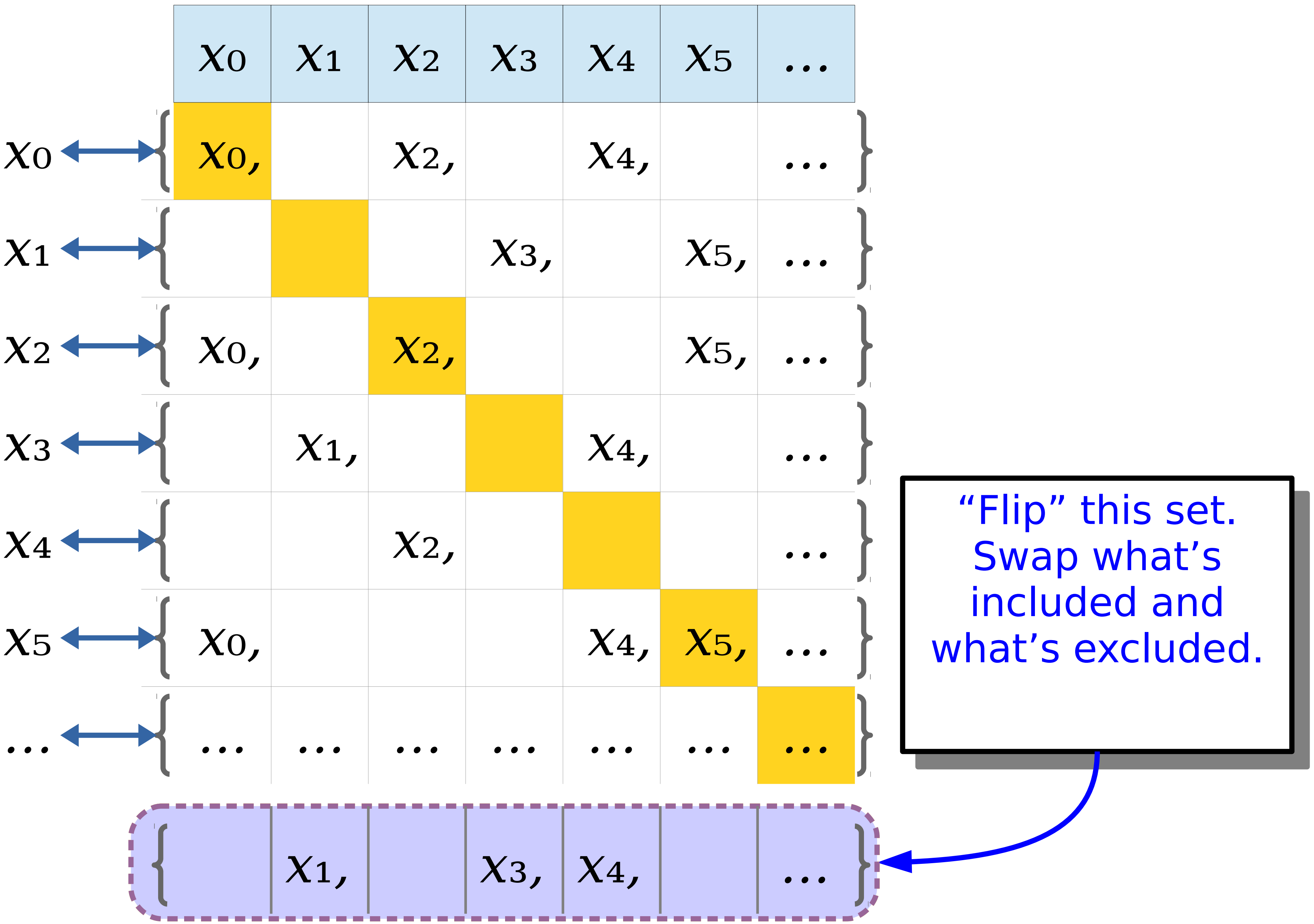
x_0 x_1 x_2 x_3 x_4 x_5 …
x_0 { x_0, x_2, x_4, … }
x_1 { x_3, x_5, … }
x_2 { x_0, x_1, x_2, x_5, … }
x_3 { x_1, x_4, … }
x_4 { x_2, … }
x_5 { x_0, x_4, x_5, … }
… { … }

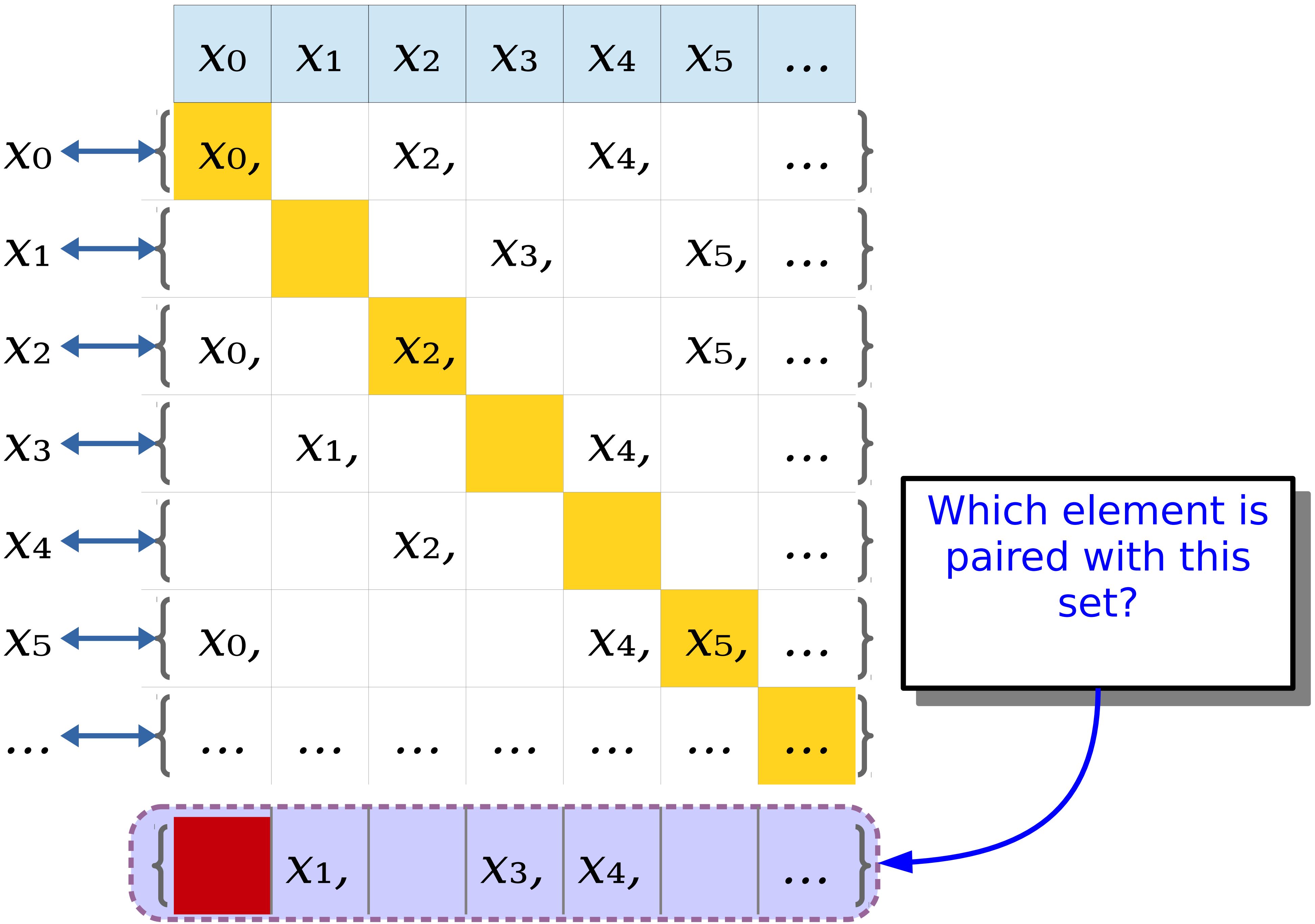
	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0 ↔	{ x_0,		x_2,		x_4,		... }
x_1 ↔	{			x_3,		x_5,	... }
x_2 ↔	{ x_0,		x_2,			x_5,	... }
x_3 ↔	{	x_1,			x_4,		... }
x_4 ↔	{		x_2,				... }
x_5 ↔	{ x_0,				x_4,	x_5,	... }
... ↔	{ ...	...	...	...	...	...	... }



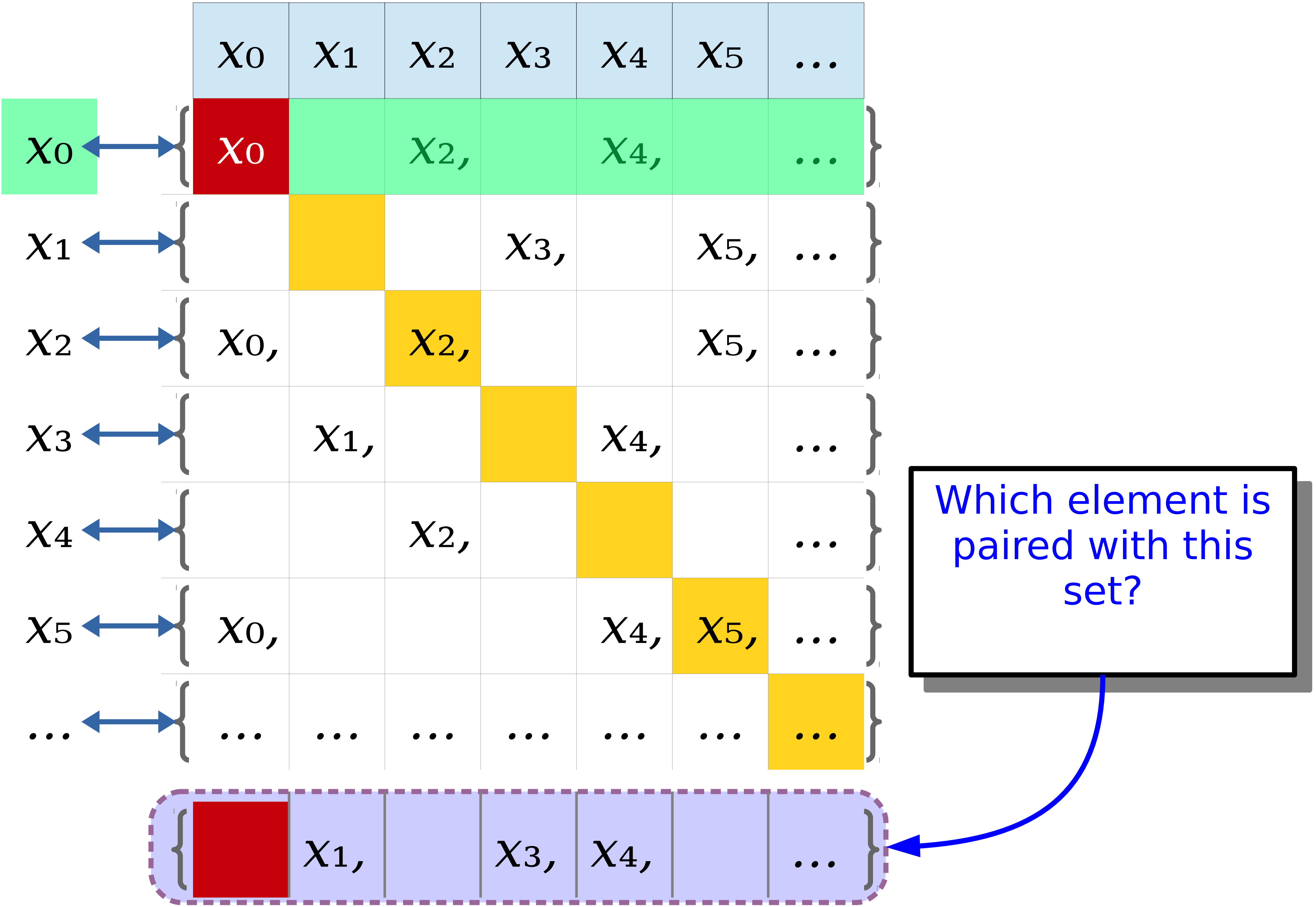
x0 x1 x2 x3 x4 x5 ...
x0 { x0, x2, x4, ... }
x1 { x3, x5, ... }
x2 { x0, x2, x5, ... }
x3 { x1, x4, ... }
x4 { x2, ... }
x5 { x0, x4, x5, ... }
... { ... ... }
{ x0, x2, x5, ... }
Which element is paired with this set?

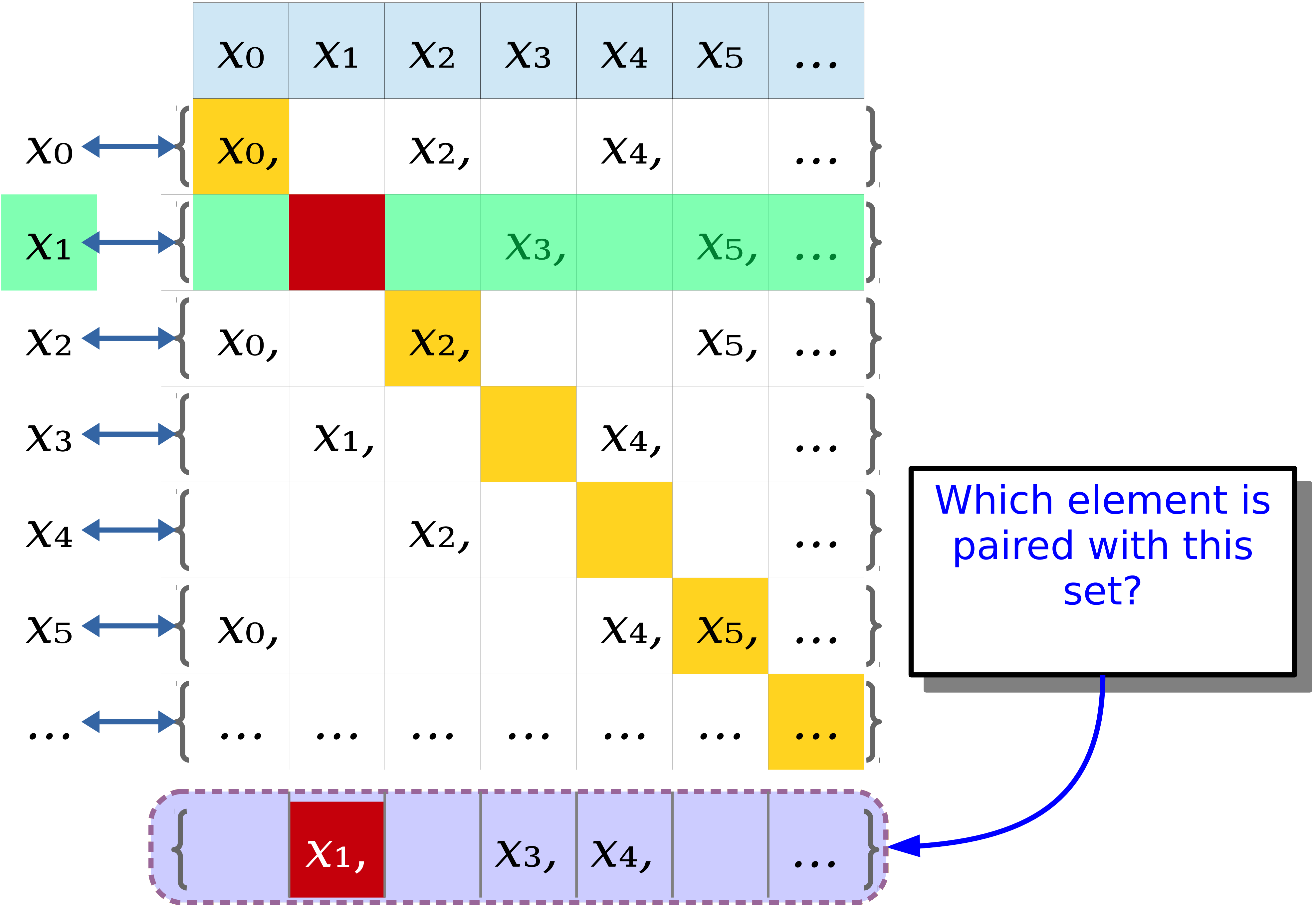

x0 x1 x2 x3 x4 x5 ...
x0 { x0, x2, x4, ... }
x1 { x3, x5, ... }
x2 { x0, x2, x5, ... }
x3 { x1, x4, ... }
x4 { x2, ... }
x5 { x0, x4, x5, ... }
... { ... ... }
{ x0, x2, x5, ... }
Which element is paired with this set?


x0 x1 x2 x3 x4 x5 ...
x0 { x0, x2, x4, ... }
x1 { x3, x5, ... }
x2 { x0, x2, x5, ... }
x3 { x1, x4, ... }
x4 { x2, ... }
x5 { x0, x4, x5, ... }
... { ... ... }
{ x1, x3, x4, ... }
“Flip” this set.
Swap what’s
included and
what’s excluded.

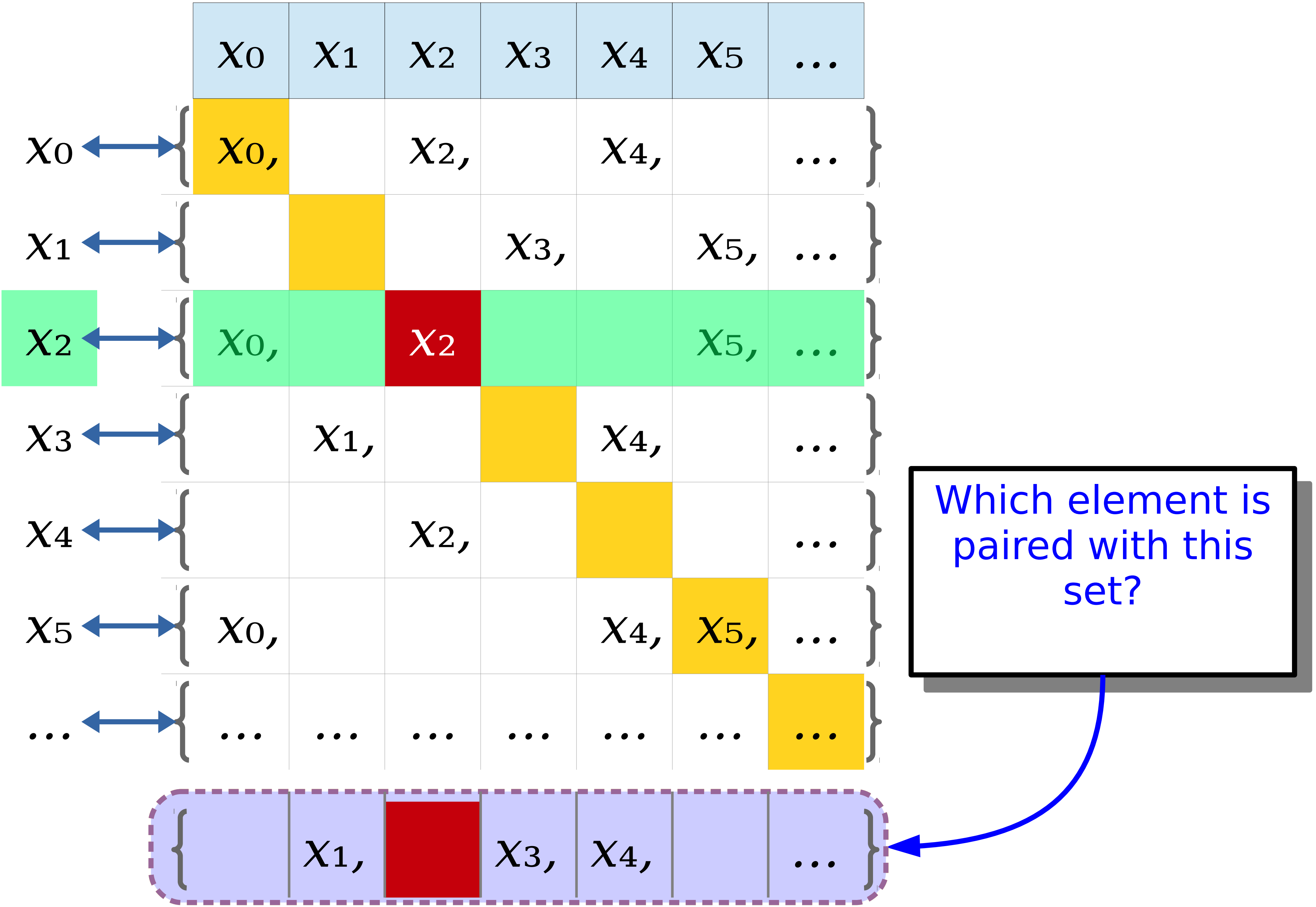


x0 x1 x2 x3 x4 x5 ...
x0 { x0, x2, x4, ... }
x1 { x3, x5, ... }
x2 { x0, x2, x5, ... }
x3 { x1, x4, ... }
x4 { x2, ... }
x5 { x0, x4, x5, ... }
... { ... ... }
{ x1, x3, x4, ... }
Which element is paired with this set?

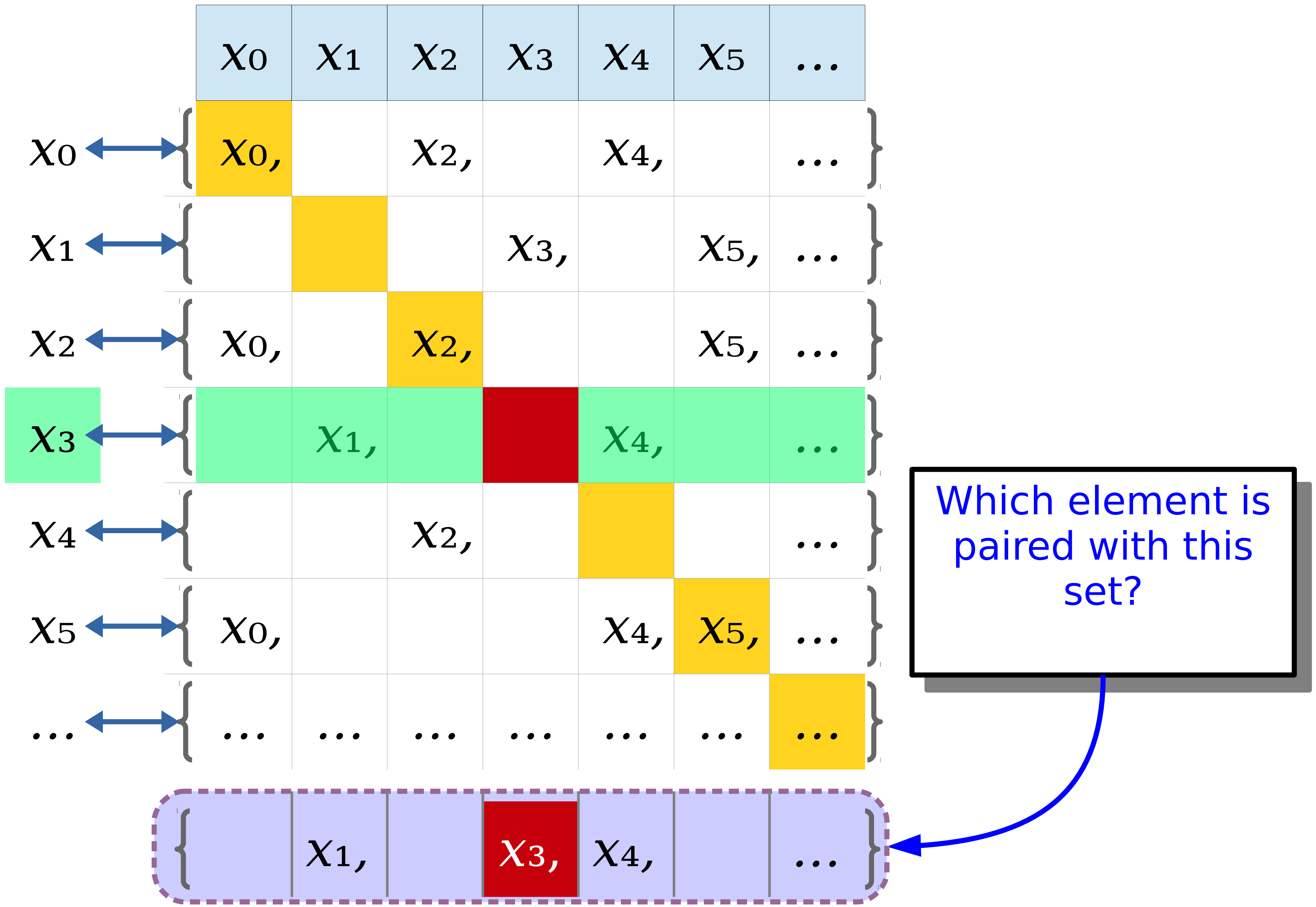

Which element is paired with this set?



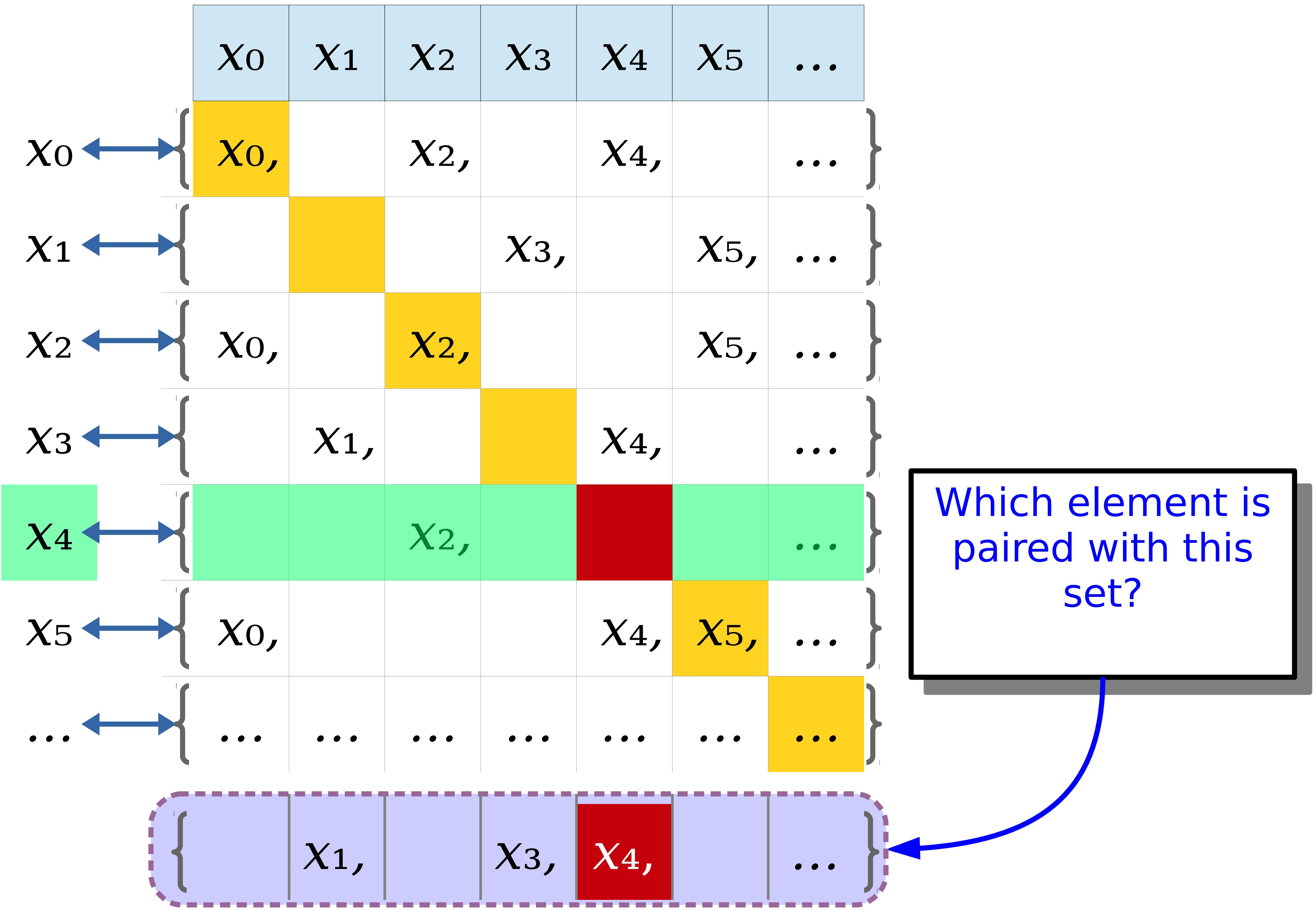
x_0 x_1 x_2 x_3 x_4 x_5 ...
x_0 { x_0, x_2, x_4, ... }
x_1 { x_3, x_5, ... }
x_2 { x_0, x_2, x_5, ... }
x_3 { x_1, x_4, ... }
x_4 { x_2, ... }
x_5 { x_0, x_4, x_5, ... }
... { ... ... }
{ x_1, x_3, x_4, ... }
Which element is paired with this set?



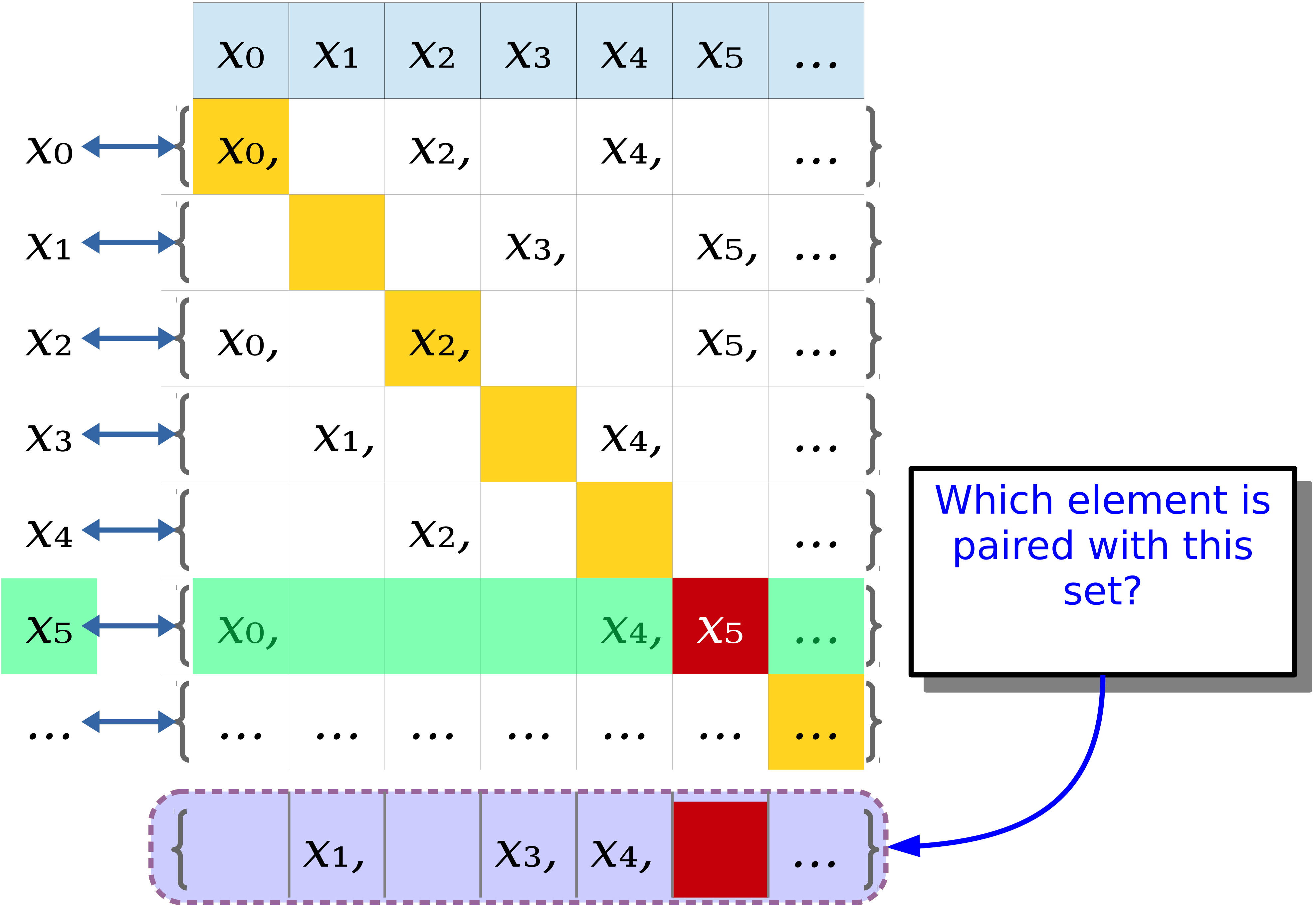
x0 x1 x2 x3 x4 x5 ...
x0 {x0, x2, x4, ...}
x1 {x3, x5, ...}
x2 {x0, x2 x5, ...}
x3 {x1, x4, ...}
x4 {x2, ...}
x5 {x0, x4, x5, ...}
... {... ... ...}
{x1, x3, x4, ...}
Which element is paired with this set?



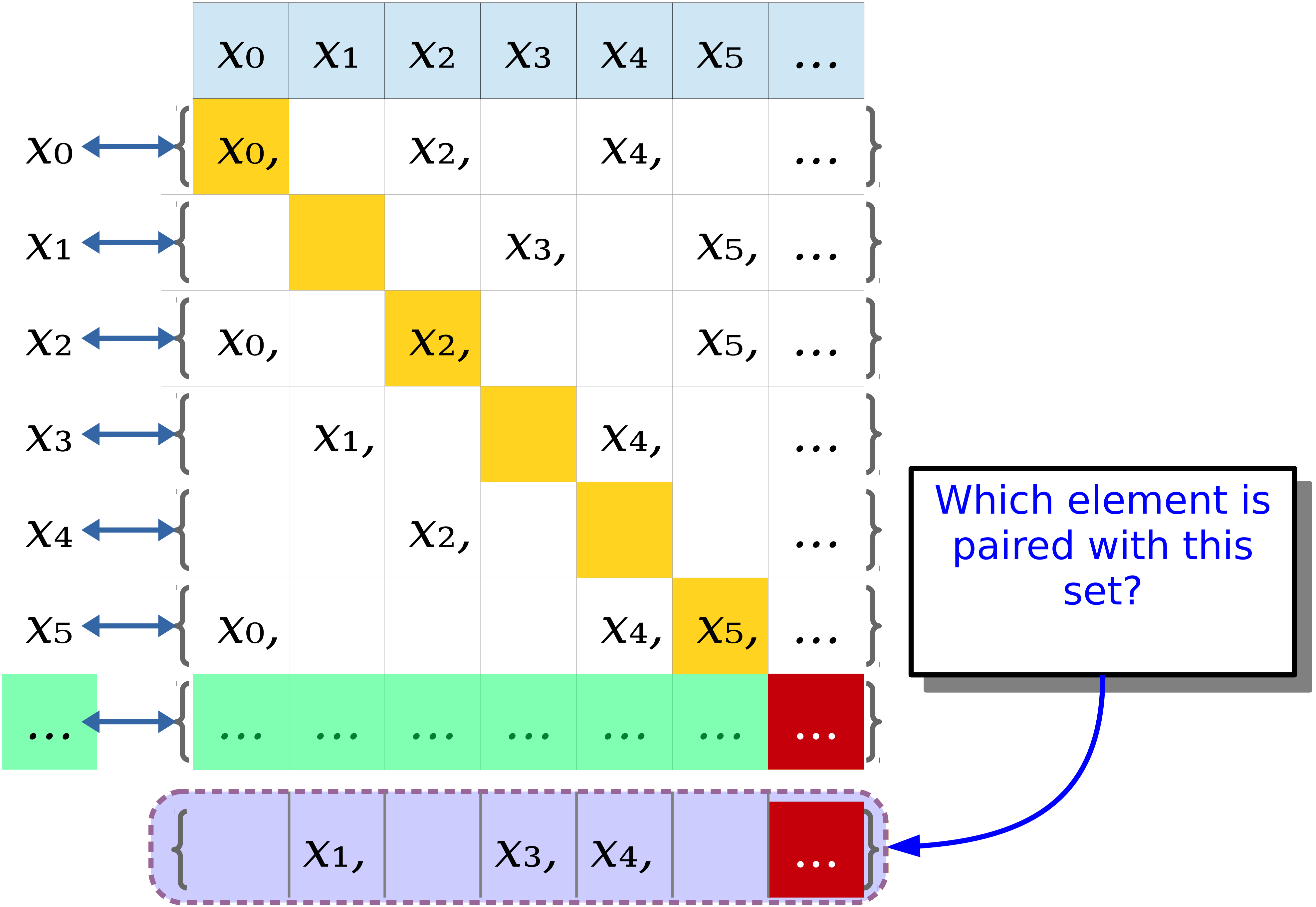
x_0 x_1 x_2 x_3 x_4 x_5 ...
x_0 { x_0, x_2, x_4, ... }
x_1 { x_3, x_5, ... }
x_2 { x_0, x_2, x_5, ... }
x_3 { x_1, x_4, ... }
x_4 { x_2, ... }
x_5 { x_0, x_4, x_5, ... }
... { ... ... }
{ x_1, x_3, x_4, ... }
Which element is paired with this set?



x0 x1 x2 x3 x4 x5 ...
x0 { x0, x2, x4, ... }
x1 { x3, x5, ... }
x2 { x0, x2, x5, ... }
x3 { x1, x4, ... }
x4 { x2, ... }
x5 { x0, x4, x5, ... }
... { ... ... }
{ x1, x3, x4, ... }
Which element is paired with this set?



x0 x1 x2 x3 x4 x5 ...
x0 { x0, x2, x4, ... }
x1 { x3, x5, ... }
x2 { x0, x2, x5, ... }
x3 { x1, x4, ... }
x4 { x2, ... }
x5 { x0, x4, x5 ... }
... { ... ... }
{ x1, x3, x4, ... }
Which element is paired with this set?



x_0 x_1 x_2 x_3 x_4 x_5 ...
x_0 { x_0, x_2, x_4, ... }
x_1 { x_3, x_5, ... }
x_2 { x_0, x_2, x_5, ... }
x_3 { x_1, x_4, ... }
x_4 { x_2, ... }
x_5 { x_0, x_4, x_5, ... }
... { ... ... }
{ x_1, x_3, x_4, ... }
Which element is paired with this set?

The Diagonalization Proof

- No matter how we pair up elements of S and subsets of S, the complemented diagonal won't appear in the table.
  - In row n, the nth element must be wrong.
- No matter how we pair up elements of S and subsets of S, there is *always* at least one subset left over.
- This result is ***Cantor's theorem***: Every set is strictly smaller than its power set:

If S is a set, then $|S| < |\wp(S)|$.

Two Infinities...

- By Cantor's Theorem:

$$|\mathbb{N}| < |\wp(\mathbb{N})|$$

...And Beyond!

- By Cantor's Theorem:

$$|\mathbb{N}| < |\wp(\mathbb{N})|$$
$$|\wp(\mathbb{N})| < |\wp(\wp(\mathbb{N}))|$$
$$|\wp(\wp(\mathbb{N}))| < |\wp(\wp(\wp(\mathbb{N})))|$$
$$|\wp(\wp(\wp(\mathbb{N})))| < |\wp(\wp(\wp(\wp(\mathbb{N}))))|$$
$$\dots$$

- ***Not all infinite sets have the same size!***
- ***There is no biggest infinity!***
- ***There are infinitely many infinities!***

What does this have to do with computation?

“The set of all computer programs”

“The set of all problems to solve”

Things on Strings

- A ***string*** is a sequence of characters.
- Two fun facts about strings:
  - There are ***at most*** as many programs as there are strings. (All programs are strings)
  - There are ***at least*** as many problems as there are sets of strings.
- There's an appendix to this slide deck that provides an overview of why these claims are true.
- These facts, plus Cantor's theorem, have terrifying implications.

Every computer program is a string.

So, the number of programs is at most the number of strings.

From Cantor's Theorem, we know that there are more sets of strings than strings.

There are at least as many problems as there are sets of strings.

$|\text{Programs}| \leq |\text{Strings}| < |\wp(\text{Strings})| \leq |\text{Problems}|$

Every computer program is a string.

So, the number of programs is at most the number of strings.

From Cantor's Theorem, we know that there are more sets of strings than strings.

There are at least as many problems as there are sets of strings.

|Programs| < |Problems|

There are more problems to solve than there are programs to solve them.

|Programs| < |Problems|

It Gets Worse

- Using more advanced set theory, we can show that there are *infinitely more* problems than solutions.
  - In fact, if you pick a totally random problem, the probability that you can solve it is *zero*.
- ***More troubling fact:*** We've just shown that *some* problems are impossible to solve with computers, but we don't know *which* problems those are!

We need to develop a more nuanced understanding of computation.

Where We're Going

- ***What makes a problem impossible to solve with computers?***
  - Is there a deep reason why certain problems can't be solved with computers, or is it completely arbitrary?
  - How do you know when you're looking at an impossible problem?
  - Are these real-world problems, or are they highly contrived?
- ***How do we know that we're right?***
  - How can we back up our pictures with rigorous proofs?
  - How do we build a mathematical framework for studying computation?

Next Time

- ***Mathematical Proof***
  - What is a mathematical proof?
  - How can we prove things with certainty?

Extra Slides

(We will revisit the diagonalization proof in more detail later in Week 4.What follows is a second example of finding two sets have equal cardinality even though their cardinality might appear different, and some additional explanation of the relationship between strings and problems.)

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$	...	-3	-2	-1	0	1	2	3	4	...

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$					0	1	2	3	4	...
	...	-3	-2	-1						

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	…
	↕	↕	↕	↕	↕	↕	↕	↕	↕	
$\mathbb{Z}$	0	1	2	3	4	5	6	7	8	…
	…	-3	-2	-1						

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	…
	↕	↕	↕	↕	↕	↕	↕	↕	↕	
$\mathbb{Z}$	0	1	2	3	4	5	6	7	8	…

… -3 -2 -1

Two sets have the same size if *there is a way* to pair their elements off without leaving any elements uncovered

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$	...	-3	-2	-1	0	1	2	3	4	...

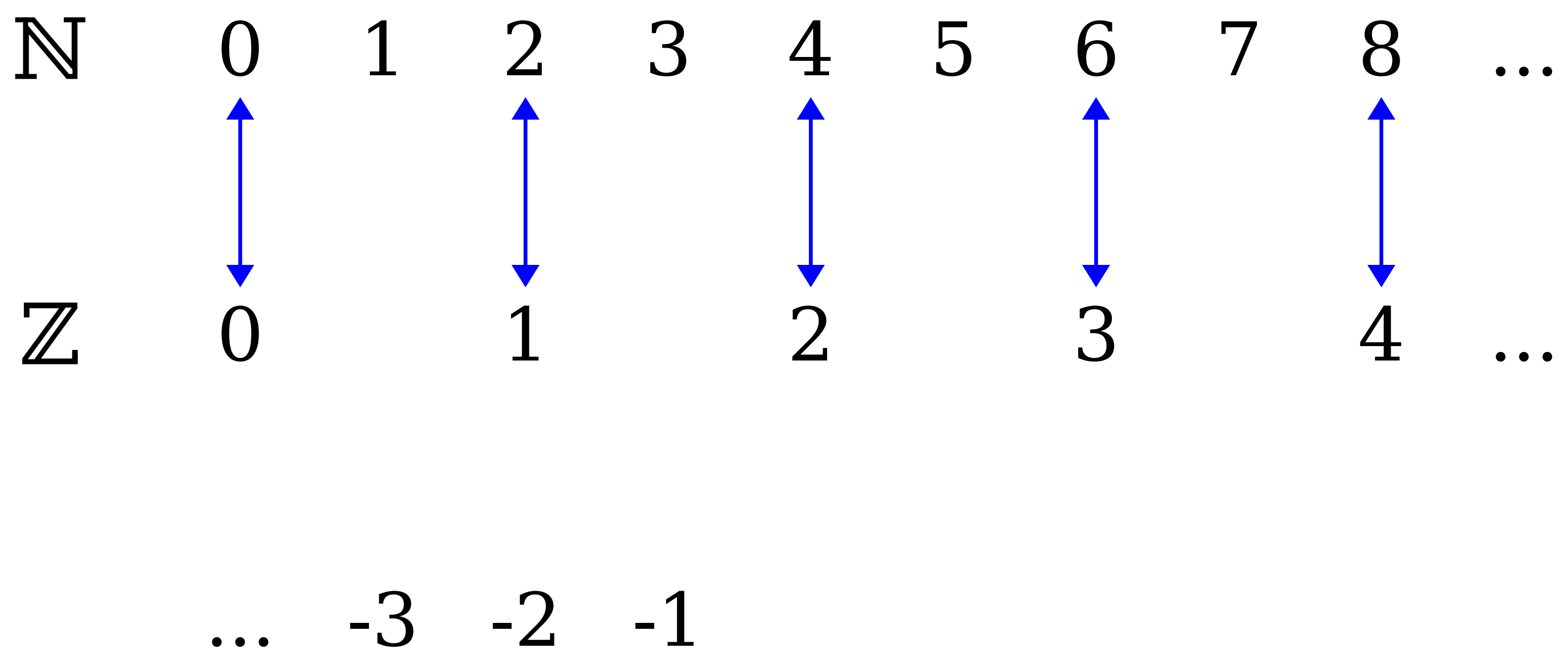
Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$										
	...	-3	-2	-1	0	1	2	3	4	...

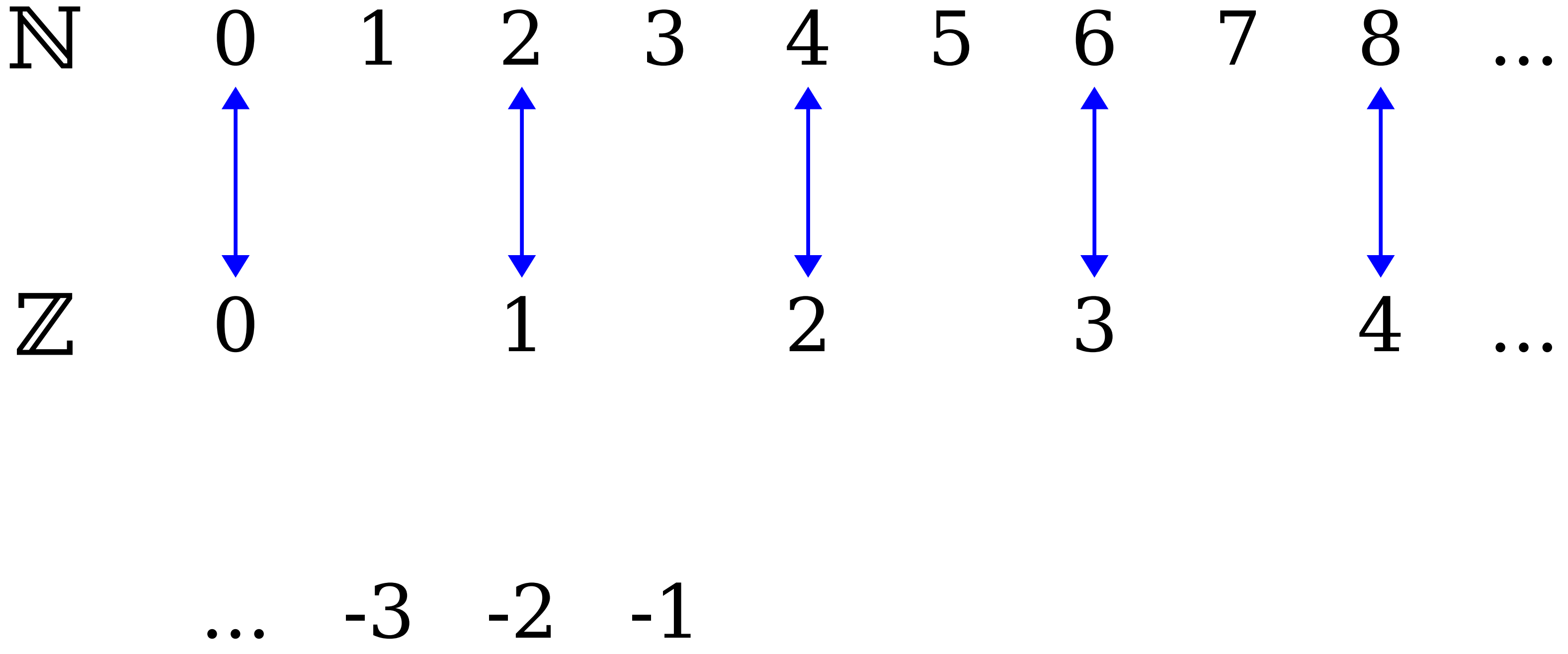
Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$	0		1		2		3		4	...
	...	-3	-2	-1						

Infinite Cardinalities



Infinite Cardinalities



Pair nonnegative integers with even natural numbers.

Infinite Cardinalities

ℕ	0	1	2	3	4	5	6	7	8	...
ℤ	0	-1	1	-2	2	-3	3	-4	4	...

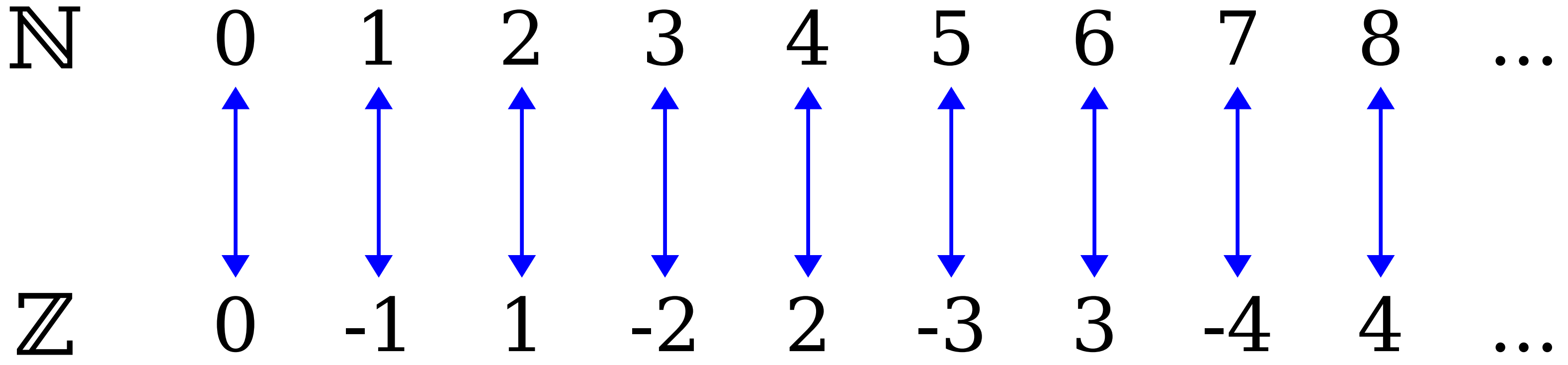
Pair nonnegative integers with even natural numbers.

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
	↕	↕	↕	↕	↕	↕	↕	↕	↕	
$\mathbb{Z}$	0	-1	1	-2	2	-3	3	-4	4	...

Pair nonnegative integers with even natural numbers.

Infinite Cardinalities



Pair nonnegative integers with even natural numbers.
Pair negative integers with odd natural numbers.

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$	0	-1	1	-2	2	-3	3	-4	4	...

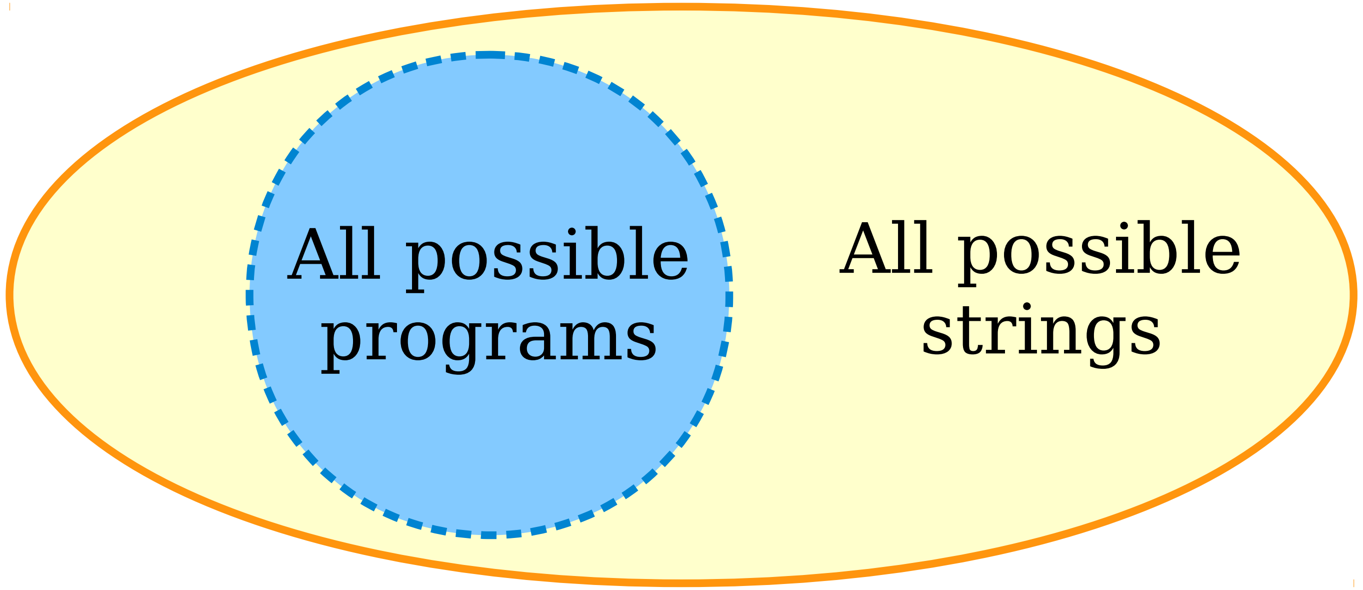
$$|\mathbb{N}| = |\mathbb{Z}| = \aleph_0$$

Pair nonnegative integers with even natural numbers.
Pair negative integers with odd natural numbers.

Appendix: String Things

Strings and Programs

- The source code of a computer program is just a (long, structured, well-commented) string of text.
- All programs are strings, but not all strings are necessarily programs.



|Programs| ≤ |Strings|

Strings and Problems

- There is a connection between the number of sets of strings and the number of problems to solve.
- Let S be any set of strings. This set S gives rise to a problem to solve:

Given a string w, determine whether $w \in S$.

Strings and Problems

Given a string *w*, determine whether $w \in S$.

- Suppose that S is the set

$$S = \{ \text{"}a\text{"}, \text{"}b\text{"}, \text{"}c\text{"}, \ldots, \text{"}z\text{"} \}$$

- From this set S, we get this problem:

Given a string *w*, determine whether *w* is a single lower-case English letter.

Strings and Problems

Given a string *w*, determine whether $w \in S$.

- Suppose that S is the set

  S = { "0", "1", "2", …, "9", "10", "11", … }

- From this set S, we get this problem:

Given a string *w*, determine whether *w* represents a natural number.

Strings and Problems

Given a string *w*, determine whether *w* ∈ *S*.

- Suppose that S is the set

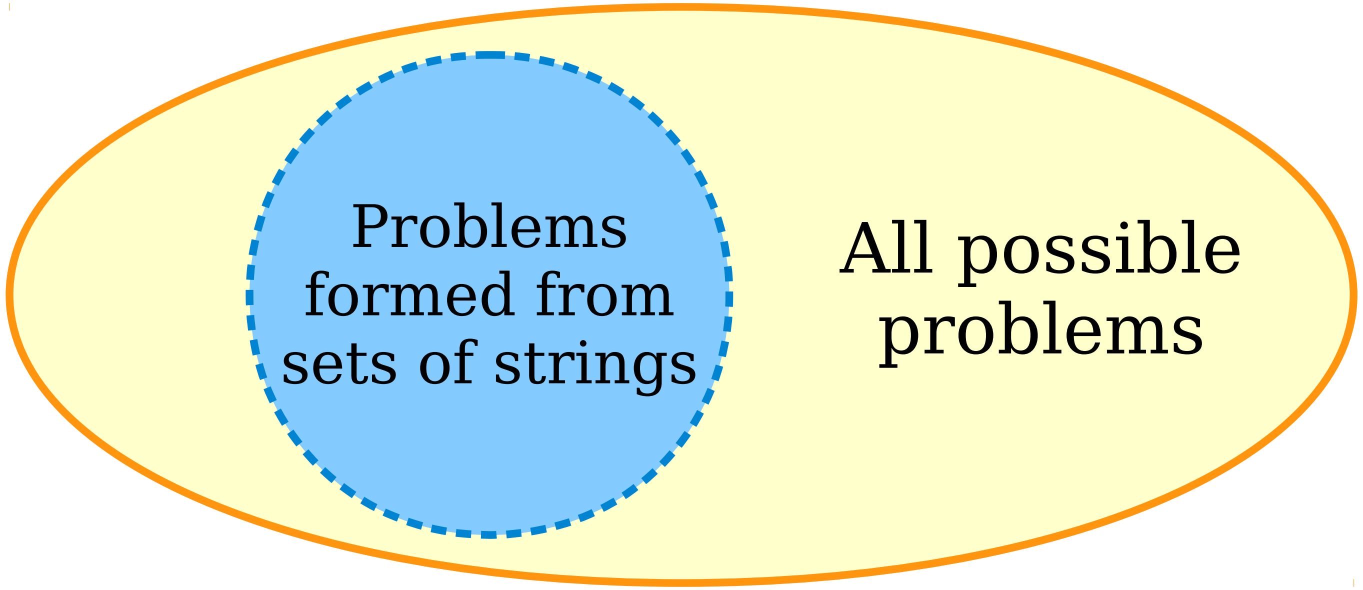
$$S = \{ \, p \mid p \text{ is a legal C++ program} \, \}$$

- From this set S, we get this problem:

Given a string *w*, determine whether *w* is a legal C++ program.

Strings and Problems

- Every set of strings gives rise to a unique problem to solve.
- Other problems exist as well.



|Sets of Strings| ≤ |Problems|